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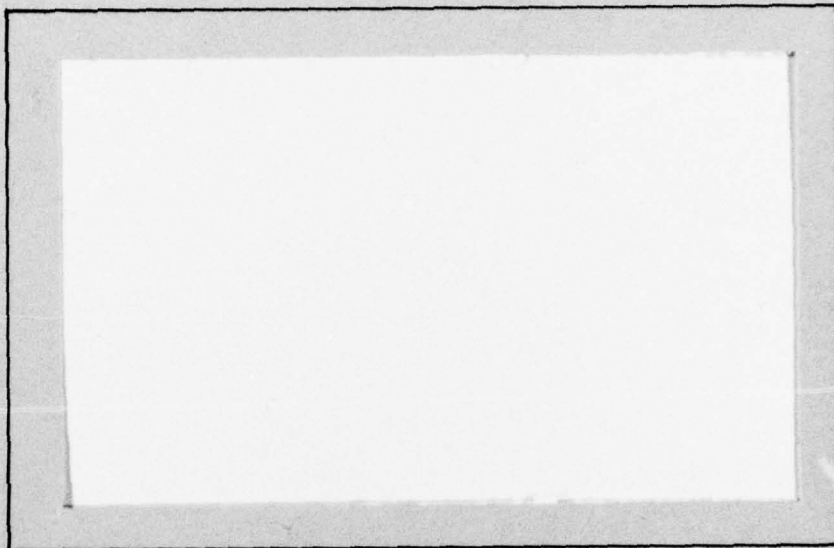


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FURTHER THEORY OF STABLE DECISIONS.

by

10 David T. Chuang

9 Technical Report No. 151

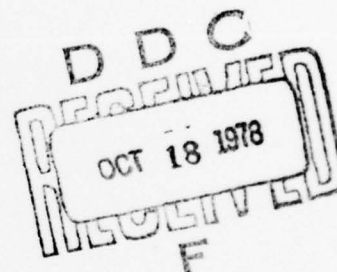
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Abstract

In a decision problem, a Bayesian has a loss function, $L_\infty(\theta, D)$ a likelihood function, $l(x|\theta)$ and a prior opinion, $F_\infty(\theta)$. A Bayesian makes a decision by minimizing expected loss. Assuming the likelihood function is fixed and agreed upon, this thesis examines the influence of small variations in loss function and prior opinion. There are a lot of ways to define small variations in loss function and prior distribution. Essentially ^{one has} ~~we have~~ to define distance between two distributions and distance between two loss functions. In this thesis we consider the distance $d(F_n, F_\infty)$ between two distributions, F_n and F_∞ , defined in such a way that $d(F_n, F_\infty)$ converges to zero iff F_n converges in distribution to F_∞ . The distance $m(L_n, L_\infty)$ between two loss functions, L_n and L_∞ , is defined so that $m(L_n, L_\infty)$ goes to zero iff L_n converges uniformly in θ and D to L_∞ . Let $D_\infty(\epsilon)$ be an ϵ -optimal decision for the triple $(L_\infty, l(x|\theta), F_\infty)$. Considering every possible sequence $F_n(\theta)$ that converges in distribution to $F_\infty(\theta)$ and every possible sequence $L_n(\theta, D)$ that converges uniformly in both θ and D to $L_\infty(\theta, D)$, this thesis examines whether $D_\infty(\epsilon)$ is still a "good" decision for the triple $(L_n, l(x|\theta), F_n)$. If every $D_\infty(\epsilon)$ is also a "good" decision for $(L_n, l(x|\theta), F_n)$ in some limiting sense, then the triple is called strongly stable. If only some specific $D_\infty(\epsilon)$ are "good" decisions for $(L_n, l(x|\theta), F_n)$ then the triple is called weakly stable, and these decisions $D_\infty(\epsilon)$ are called stabilizing decisions. If a triple is neither strongly stable nor weakly stable, it is called unstable.

Our purpose is to examine every possible triple $(L_\infty, \ell(\underline{x}|\theta), F_\infty)$ and see whether it is strongly stable, weakly stable or unstable. This thesis studies the one dimensional estimation problem, i.e., $L_\infty(\theta, D) = h(\theta - D)$, where h is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. We assume the likelihood function $\ell(\underline{x}|\theta)$ as a function of θ is continuous and uniformly bounded. Under these conditions, necessary and sufficient conditions for a triple to be strongly stable, weakly stable or unstable are available.

Later it is assumed $\ell(\underline{x}|\theta) \equiv 1$, i.e. the problem studied in Kadane and Chuang [1978]. Then, in the estimation problem, whether a triple is stable or not depends on the loss function only, and a triple can only be strongly stable or unstable. Necessary and sufficient conditions for a triple to be stable are much simpler than the conditions for the general likelihood function. Squared error loss is unstable with any opinion if the parameter space is the real line, however, absolute error loss is stable. Several other examples are also given to show how to apply these necessary and sufficient conditions.

Acknowledgements

It is a pleasure to thank Professor Joseph B. Kadane, my advisor, for suggesting this problem to me. His continuing interest and encouragement makes this research very enjoyable. I would also like to thank my thesis committee, Professors W. W. Davis, M. H. DeGroot, D. Lambert and E. C. Prescott for their comments on my thesis. I am particularly grateful to Professor DeGroot for introducing me to the realm of Statistics.

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Further Theory of Stable Decisions

Chapter 1. Introduction

1.1 Formulation of the Problem

The original results about stable decision problems were introduced in Kadane and Chuang [1978]. As stated in that paper, "Generally when a personalistic Bayesian tells you his loss function and opinion, he means them only approximately. He hopes that his approximation is good, and that whatever errors he may have made will not lead to decisions with loss substantially greater than he would have obtained had he been able to write down his true loss function and opinion." What kinds of loss functions and opinions satisfy these conditions are the objects of our study.

As in that paper, "there are two special cases that have been considered. In the first, we cannot (or need not) obtain one's exact prior probability. Stone [1963] studied decision procedures with respect to the use of wrong prior distributions. He emphasized the possible usefulness of non-ideal procedures that do not require full specification of the prior probability distribution.

Fishburn, Murphy and Isaacs [1967] and Pierce and Folks [1969] also discussed decision making under uncertainty when the decision maker has difficulty in assigning prior probabilities. They outlined six approaches that may be used to assign probabilities. In the second case, one cannot obtain one's exact utility function. Britney and Winkler [1974] have investigated the properties of Bayesian point estimates under loss functions other than the simple linear and quadratic loss functions. They also discussed the sensitivity of Bayesian point estimates to misspecification in the

loss function. Schlaiffer [1959] and Antelman [1965] discuss relating the utility of the optimal decision to the utility of suboptimal decisions in certain contexts.

The closest related work, however, is the material on stable estimation in Edwards, Lindman and Savage [1963]. They propose that there is data such that the likelihood function will be sufficiently peaked as to dominate the prior distribution. The criterion for robustness is that the densities of various possible posterior distributions are close.

Another important line of comparison is the work on robustness in the classical context as exemplified for instance, in Andrews et al. [1972], Bickel and Lehmann [1975 a,b] and Huber [1972,1973]. While they study how estimates change as a consequence of outliers, we study here how the worth of the estimates change."

In the same paper, we have the following formalizations and definitions. Suppose that the parameter space is $\Theta \subset \mathbb{R}^k$ for some k , and the decision space is $\mathcal{D} \subset \mathbb{R}^l$ for some l . If $F_\infty(\theta)$ is my (approximate) opinion over $\theta \in \Theta$, and $L_\infty(\theta, D)$ my (approximate) loss function, the (approximate) loss of the decision problem to me is

$$(1.1) \quad W_\infty = \inf_{D \in \mathcal{D}} \int L_\infty(\theta, D) dF_\infty(\theta)$$

which is here assumed to be finite. Then for every $\epsilon > 0$, there is a decision $D_\infty(\epsilon)$ which is ϵ -optimal, that is

$$(1.2) \quad \int L_\infty(\theta, D_\infty(\epsilon)) dF_\infty(\theta) \leq W_\infty + \epsilon.$$

Suppose, however, that my "true" opinion over Θ is close to $F_\infty(\theta)$ and my "true" loss function over $\Theta \times \mathcal{D}$ is close to $L_\infty(\theta, D)$. Now we have to define the term "close." In order to do this we need distance function between two functions. There are many ways to define distance between two distribution functions and between two loss functions. In our problem we define the distance $d(F_n, F_\infty)$ between $F_n(\theta)$ and $F_\infty(\theta)$ in such a way that $d(F_n, F_\infty)$ converges to zero iff F_n converges weakly to F_∞ , and we define the distance $m(L_n, L_\infty)$ between $L_n(\theta, D)$ and $L_\infty(\theta, D)$ in such a way that $m(L_n, L_\infty)$ converges to zero iff L_n converges uniformly in θ and D to L_∞ . These distance functions are not uniquely defined, but all the results in this thesis will hold for any functions satisfying the above conditions.

For any fixed $D_\infty(\epsilon)$, let $B_n(\epsilon)$ be defined by

$$(1.3) \quad B_n(\epsilon) = \int L_n(\theta, D_\infty(\epsilon)) dF_n(\theta) - \inf_D \int L_n(\theta, D) dF_n(\theta).$$

We are interested in finding the limit, $\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} B_n(\epsilon)$.

In this thesis, $\varlimsup_{n \rightarrow \infty}$ is represented as $\limsup_{n \rightarrow +\infty}$.

Definition 1: A pair $(L_\infty(\theta, D), F_\infty(\theta))$ is called strongly stable, if for every

$$F_n \xrightarrow{W} F_\infty, \quad L_n \rightarrow L_\infty \quad \text{uniformly in } \theta, D, \text{ and}$$

for every choice of $D_\infty(\epsilon)$ satisfying (1.2),

$$(1.4) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} B_n(\epsilon) = 0.$$

In this case, the pair (L_∞, F_∞) is called strongly stable. There are situations in which (1.4) holds for every choice of $L_n \rightarrow L_\infty$ and $F_n \rightarrow F_\infty$, but only for some particular choice $D_\infty(\epsilon)$.

In this case, $D_\infty(\epsilon)$ is called the stabilizing decision, and the pair (L_∞, F_∞) is called weakly stable. If (L_∞, F_∞) is neither strongly nor weakly stable, it is called unstable.

The motivation for these definitions is that if a pair (L_∞, F_∞) is strongly stable, then small errors in $L_\infty(\theta, D)$ or $F_\infty(\theta)$ will not result in substantially worse decisions. If on the other hand, a Bayesian finds that the loss function and opinion he has written down are unstable, then he may wish to reassess his loss function and opinion to be certain that no errors have been made. When he finds he has written down a loss function and opinion which is weakly but not strongly stable, a Bayesian may choose to make the stabilizing decision to have protection against errors in either the loss function or opinion.

From a more general point of view we can formulate our problem as follows:

Definition 2: Let $L_n \rightarrow L_\infty$ and $W_n \rightarrow L_\infty$ uniformly in θ, D , and $F_n \xrightarrow{W} F_\infty, G_n \xrightarrow{W} F_\infty$.

Let $D_n(\epsilon)$ satisfy

$$(1.5) \quad \int W_n(\theta, D_n(\epsilon)) dG_n(\theta) \leq \inf_D \int W_n(\theta, D) dG_n(\theta) + \epsilon.$$

If for every such choice of $D_n(\epsilon)$,

$$(1.6) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [\int L_n(\theta, D_n(\epsilon)) dF_n(\theta) - \inf_D \int L_n(\theta, D) dF_n(\theta)] = 0,$$

then (L_∞, F_∞) is strongly stable. If there is some choice of $D_n(\epsilon)$ which makes (1.6) hold, then (L_∞, F_∞) is weakly stable and $D_n(\epsilon)$ is the stabilizing decision. If a pair (L_∞, F_∞) is neither strongly nor weakly stable, it is called unstable.

The following two definitions, definitions 3 and 4, can be shown equivalent to definitions 1 and 2, respectively. They are easier to handle and simplify our problem.

Definition 3: For any $\epsilon > 0$, define $D_\infty(\epsilon)$ as in (1.2). Then (L_∞, F_∞) is strongly (weakly) stable iff for every sequence $F_n \xrightarrow{W} F_\infty$ and for every (for some) such $D_\infty(\epsilon)$,

$$(1.7) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [\int L_\infty(\theta, D_\infty(\epsilon)) dF_n(\theta) - \inf_D \int L_\infty(\theta, D) dF_n(\theta)] = 0.$$

Definition 4: For any $\epsilon > 0$, define the decision $D_n(\epsilon)$ as in (1.5) but with W_n taken to be L_∞ . Then (L_∞, F_∞) is strongly (weakly) stable iff for every sequence $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$, and for every (for some) such $D_n(\epsilon)$, (1.6) holds with L_∞ substituted for L_n .

Thus definitions 3 and 4 differ from definitions 1 and 2 in that, for the latter, only the opinions move, while the loss functions stay constant.

Under the above definitions, the following results are also available in Kadane and Chuang [1978].

Theorem 1.1: (a) (L_∞, F_∞) is strongly (weakly) stable by definition 1 iff (L_∞, F_∞) is strongly (weakly) stable by definition 3.

(b) (L_∞, F_∞) is strongly (weakly) stable by definition 2 iff (L_∞, F_∞) is strongly (weakly) stable by definition 4.

A function $L(\theta, D)$ is called continuous in θ uniformly in D iff for all $\epsilon > 0$ and θ , there exists $\delta > 0$ such that for all D , $|\theta - \theta_0| < \delta$ implies $|L(\theta, D) - L(\theta_0, D)| < \epsilon$.

Under this definition, we have:

Theorem 1.2: Suppose (i) $|L_\infty(\theta, D)| \leq B$ for all θ and D
(ii) $L_\infty(\theta, D)$ is continuous in θ uniformly in D . Then (L_∞, F_∞)
is strongly stable by definitions 1, 2, 3 and 4.

We also know that squared error loss is unstable with any opinion if the parameter space is the real line. Under the same conditions, it can be shown that absolute loss is strongly stable with any opinion. Later in that paper, we examine the estimation case, i.e. $L_\infty(\theta, D) = h(\theta - D)$, where $h(x)$ is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. While these conditions are not enough to assure stability, several conditions are given that are sufficient.

There are, as stated in Kadane and Chuang [1978], a number of interesting and potentially enlightening choices that might be made for the sense of convergence of F_n to F_∞ and L_n to L_∞ . We start with weak convergence in the distribution and uniform convergence in both arguments in the losses. Whether our results still hold under different senses of convergence of F_n to F_∞ and L_n to L_∞ , is an interesting problem for further research.

In this thesis, we restrict ourselves to one-dimensional problems. Thus we assume the parameter space is $\Theta \subset \mathbb{R}^1$ and the decision space is $\mathcal{D} \subseteq \mathbb{R}^1$. Suppose X_1, X_2, \dots, X_m are random variables whose joint distribution depends on the value of θ . For the given value x_1, \dots, x_m , of the distributions, let $\underline{x} = (x_1, x_2, \dots, x_m)$. Then we have likelihood function $l(\underline{x}|\theta)$,

which is assumed to be a continuous and uniformly bounded function of θ . Let

$$P(\theta) = \frac{\int_{-\infty}^{\theta} \ell(\underline{x}|t) dF(t)}{\int_{-\infty}^{\infty} \ell(\underline{x}|t) dF(t)}, \text{ so that } P(\theta) \text{ is the posterior distribution}$$

of θ corresponding to the prior distribution $F(\theta)$ and the likelihood function $\ell(\underline{x}|\theta)$. After \underline{x} has been observed, the decision problem is basically the same as before observing \underline{x} , except the distribution of θ has changed from the prior to the posterior distribution.

Let $P_{\infty}(\theta)$ denote the posterior distribution of θ corresponding to the prior distribution $F_{\infty}(\theta)$ and likelihood function $\ell(\underline{x}|\theta)$. Also let $P_n(\theta)$ be the posterior distribution of θ corresponding to the prior distribution $F_n(\theta)$ and the likelihood function $\ell(\underline{x}|\theta)$. We then have the following definitions.

Definition I: Let $F_n \xrightarrow{W} F_{\infty}$, $L_n(\theta, D) \rightarrow L_{\infty}(\theta, D)$ uniformly in θ and D , and let $D_{\infty}(\epsilon)$ be a decision satisfying

$$(1.8) \quad \int L_{\infty}(\theta, D_{\infty}(\epsilon)) dP_{\infty}(\theta) \leq \inf_D \int L_{\infty}(\theta, D) dP_{\infty}(\theta) + \epsilon.$$

Now let $B_n(\epsilon)$ be defined by

$$(1.9) \quad B_n(\epsilon) = \int L_n(\theta, D_{\infty}(\epsilon)) dP_n(\theta) - \inf_D \int L_n(\theta, D) dP_n(\theta).$$

If

$$(1.10) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} B_n(\epsilon) = 0$$

for every choice of $L_n \rightarrow L_{\infty}$ uniformly in θ and D , and $F_n \xrightarrow{W} F_{\infty}$ and every choice of $D_{\infty}(\epsilon)$ satisfying (1.8), then the triple $(L_{\infty}, \ell(\underline{x}|\theta), F_{\infty})$ is called strongly stable. There are cases in which

(1.10) holds for every choice of $L_n \rightarrow L_\infty$ and $F_n \rightarrow F_\infty$, but only for some particular choice of $D_\infty(\epsilon)$. In this case, $D_\infty(\epsilon)$ is called a stabilizing decision and the triple $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is called weakly stable. If the triple is not stable (either strongly or weakly), it is called unstable.

Definition II: Let $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$; $W_n \rightarrow L_\infty$ and $L_n \rightarrow L_\infty$ uniformly in θ and D , and let $D_n(\epsilon)$ be a decision satisfying

$$(1.11) \quad \int W_n(\theta, D_n(\epsilon)) dQ_n(\theta) \leq \inf_D \int W_n(\theta, D) dQ_n(\theta) + \epsilon$$

where $Q_n(\theta)$ is the posterior distribution of θ corresponding to the prior distribution $G_n(\theta)$ and the likelihood function $l(\underline{x}|\theta)$.

Also define $C_n(\epsilon)$ as follows:

$$(1.12) \quad C_n(\epsilon) = \int L_n(\theta, D_n(\epsilon)) dP_n(\theta) - \inf_D \int L_n(\theta, D) dP_n(\theta).$$

If for every choice of F_n , G_n , L_n , W_n and $D_n(\epsilon)$

$$(1.13) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} C_n(\epsilon) = 0$$

then the triple $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is called strongly stable. If there is some choice of $D_n(\epsilon)$ which makes (1.13) hold, then $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is called weakly stable and $D_n(\epsilon)$ is called the stabilizing decision. If a triple is neither strongly stable nor weakly stable, it is called unstable.

Definition III is similar to definition I, except we let $L_n = L_\infty$ for all n . Also definition IV is similar to definition II, except, for all n , we let $W_n = L_n = L_\infty$.

One related work is Shapiro [1975]. She considers the case that the prior distribution is known exactly and likelihood function may have small variation from the "correct" one. She found sufficient conditions that the posterior distributions will converge in distribution to the "correct" posterior distribution.

Rubin [1977] studies robust Bayesian estimation by assuming that the observation X is normal with mean θ and variance 1, and that θ has a prior distribution, which is chosen to be normal, double-exponential, logistic, or Cauchy with mean 0. He then investigates the behavior of Bayes estimates when the true prior distribution and assumed prior distribution are different members of this set of distributions.

DeRobertis [1978] studies and develops betting sets such that prior beliefs are quantified by the acceptance of a sub-family of the available bets. From the bets accepted by a decision maker we generate a class of prior distributions. We then can do posterior inferences and study convergence of opinion as evidence accumulates.

1.2 Explanation of Definitions

In this thesis we introduce twelve definitions. They can be divided into three groups: Definitions I, II, III and IV are in the first group, definitions I', II', III' and IV' are in the second group and definitions 1, 2, 3 and 4 are in the third group. Definitions I, II, III and IV are the primary interest of the thesis. When the likelihood function satisfies $l(\underline{x}|\theta) \equiv 1$ then definitions I, II, III and IV reduce to definitions 1, 2, 3 and 4 respectively. Thus definitions in the third group are just special cases of definitions in the first group. In Chapter 5 we discuss special properties of definitions and triples in the third group. The reason we introduce definitions I', II', III' and IV' is that they are easier to handle mathematically. Theorem 2.1 shows definition I', II', III' and IV' are equivalent to definitions I, II, III and IV respectively.

We can see definitions I, II, III and IV from a more general point of view. Suppose d^1 and d^2 are arbitrary metrics on measures, and m^1 and m^2 are arbitrary metrics on loss functions. Let $d^1(F_n, F_\infty) + d^2(G_n, F_\infty) + m^1(L_n, L_\infty) + m^2(W_n, L_\infty) \rightarrow 0$. Consider the following special metrics: let $d_1(F, G)$ be the Levy Distance between two distributions F and G . Let $d_2(F, G) = 0$ if $F \equiv G$ and 1 otherwise. Also let $m_1(L_1, L_2) = \sup_{\theta, D} |L_1(\theta, D) - L_2(\theta, D)|$ and let $m_2(L_1, L_2) = 0$ if $L_1 \equiv L_2$ and 1 otherwise. Then we have the following relations among the definitions.

Definition	d^1	d^2	m^1	m^2
I	d_1	d_2	m_1	m_2
II	d_1	d_1	m_1	m_1
III	d_1	d_2	m_2	m_2
IV	d_1	d_1	m_2	m_2 .

In each group of definitions, it can be shown that the first one is equivalent to the third one and the second one is equivalent to the fourth one. However, the first and the second definitions in the same group are not necessarily equivalent for all triples. In this thesis we are particularly interested in the estimation problem, i.e. $L_\infty(\theta, D) = h(\theta - D)$, where h is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. Under these conditions all definitions in the same group are equivalent to each other (as shown in Chapter 3). Thus for the estimation problem, definitions I, II, III, IV, I', II', III' and IV' are all equivalent to each other. Also definitions 1, 2, 3 and 4 are equivalent to each other.

1.3 Outline of Thesis

In Chapter 2, we study the general structure of definitions I, II, III and IV. We introduce definitions I', II', III' and IV' that are equivalent to definitions I, II, III and IV respectively. However, they are easier to handle mathematically. Later we show $P_n(\theta) \xrightarrow{W} P_\infty(\theta)$, if $F_n(\theta) \xrightarrow{W} F_\infty(\theta)$, where $P_n(\theta)(P_\infty(\theta))$ is the posterior distribution corresponding to prior distribution $F_n(\theta)$ ($F_\infty(\theta)$) and the likelihood function $\ell(\underline{x}|\theta)$.

In Chapter 3, we study relationships among definitions I, II, III and IV. Definitions I and II are equivalent to definitions III and IV respectively. However, by an example, we show that definitions I and II are not equivalent for every triple. Then we study the estimation problem, i.e. $L_\infty(\theta, D) = h(\theta - D)$, where h is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. Under these conditions, definitions I, II, III and IV are equivalent. Thus definitions I, II, III, IV, I', II', III' and IV' are equivalent to each other in this case, and it follows directly that definitions 1, 2, 3 and 4 are equivalent to each other.

In Chapter 4, we study the prediction problem in detail. Our object is to determine whether a triple is strongly stable, weakly stable or unstable. Necessary and sufficient conditions for a triple to be strongly stable, weakly stable or unstable are proved.

In Chapter 5, we reexamine the case $\ell(\underline{x}|\theta) \equiv 1$, considered in Kadane and Chuang [1978]. Again we study the estimation problem. We have some new results. It is impossible

for a triple to be weakly stable. For any two distributions F_∞ and G_∞ and any loss function $h(\theta-D)$, (h, F_∞) is strongly stable iff (h, G_∞) is strongly stable. Thus, whether (h, F_∞) is stable or not depends on h only. Necessary and sufficient conditions for loss function to be stable are given. They are much simpler and easier to check than the conditions in Chapter 4. By using these conditions, we reexamine the examples in Kadane and Chuang [1978].

Chapter 2 A General Structure and Convergence of $P_n(\theta)$

2.1 A General Structure and New Definitions

The object of this section is to simplify our mathematical formulation. We show that we can let the denominator of the posterior distribution equal 1 without loss of generality. Then we introduce definitions I', II', III' and IV' that are equivalent to definitions I, II, III and IV respectively. The advantage of these new definitions is that they are easier to handle mathematically. Finally we show $P_n(\theta) \xrightarrow{W} P_\infty(\theta)$, if $F_n(\theta) \xrightarrow{W} F_\infty(\theta)$, where $P_n(\theta)$ ($P_\infty(\theta)$) is the posterior distribution corresponding to prior distribution $F_n(\theta)$ ($F_\infty(\theta)$) and the likelihood function $l(\underline{x}|\theta)$. This result connects definitions I, II, III and IV to definitions 1, 2, 3 and 4. We discuss its importance in Chapter 5.

In the following we show, by using definition II, the denominator of the posterior distribution will not affect the stability of a triple. By letting $G_n = F_\infty$, or $G_n = F_\infty$ and $W_n = L_n = L_\infty$, or $W_n = L_n = L_\infty$, we can see the same result holds for definitions I, III and IV respectively.

We assume $\int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_\infty(\theta) = C > 0$. If $C = 0$, then we put all of our prior probability on the set of θ that has $l(\underline{x}|\theta) = 0$. This is not the case we are interested in. We keep this assumption throughout the thesis.

Lemma 2.1. Consider the triple $(L_\infty, l(\underline{x}|\theta), F_\infty)$, and suppose $\int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_\infty(\theta) = C > 0$. Suppose $L_n \rightarrow L_\infty$ and $W_n \rightarrow L_\infty$ uniformly in θ, D ; also suppose $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$.

Let $D'_n(\epsilon')$ be any decision satisfying

$$(2.1) \quad \int W_n(\theta, D'_n(\epsilon')) \ell(\underline{x}|\theta) dG_n(\theta) \leq \inf_D \int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) + \epsilon'$$

and let $D_n(\epsilon)$ be any decision satisfying

$$(2.2) \quad \frac{\int W_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} \leq \inf_D \frac{\int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} + \epsilon.$$

And let $\{D_\infty(\epsilon)\}$ be the set of all decisions satisfying (2.2).

Let $C'_n(\epsilon')$ and $C_n(\epsilon)$ be defined by

$$(2.3) \quad C'_n(\epsilon') = \int L_n(\theta, D'_n(\epsilon')) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_n(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta)$$

and

$$(2.4) \quad C_n(\epsilon) = \frac{\int L_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta)}{\int \ell(\underline{x}|\theta) dF_n(\theta)} - \inf_D \frac{\int L_n(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta)}{\int \ell(\underline{x}|\theta) dF_n(\theta)}.$$

Then (a) for all $D_n(\epsilon)$, we have $\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} C_n(\epsilon) = 0$ iff
for all $D'_n(\epsilon')$, $\lim_{\epsilon' \downarrow 0} \limsup_{n \rightarrow \infty} [C'_n(\epsilon')] = 0$;

(b) there exists $D_n(\epsilon)$, such that $\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} C_n(\epsilon) = 0$ iff
there exists $D'_n(\epsilon')$ satisfying $\lim_{\epsilon' \downarrow 0} \limsup_{n \rightarrow \infty} [C'_n(\epsilon')] = 0$.

Proof: (a) Since $\ell(\underline{x}|\theta)$ is continuous and uniformly bounded, by the Helly-Bray Theorem (Loeve [1963] page 181), we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dF_n(\theta) = \int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dF_\infty(\theta) = C$$

and

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dG_n(\theta) = \int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dF_\infty(\theta) = C.$$

Then there exists an N , such that for all $n > N$

$$(2.5) \quad \frac{C}{2} < \int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dF_n(\theta) < \frac{3}{2} C$$

and

$$(2.6) \quad \frac{C}{2} < \int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dG_n(\theta) < \frac{3}{2} C.$$

By assumption, for all $D_n(\epsilon)$, we have $\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [C_n(\epsilon)] = 0$

The strategy is to find a proportional constant α , such that when $\epsilon' = \alpha\epsilon$, then every $D'_n(\epsilon')$ satisfies (2.2). Then, by assumption, it is straightforward to show $\lim_{\epsilon' \downarrow 0} \limsup_{n \rightarrow \infty} [C'_n(\epsilon')] = 0$.

We know $D'_n(\epsilon')$ satisfies (2.1). Let

$$(2.7) \quad \epsilon' = \frac{1}{2} C\epsilon.$$

Then for all $n > N$ and all $D'_n(\epsilon')$ we have

$$\begin{aligned} & \frac{\int W_n(\theta, D'_n(\epsilon')) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} \\ & \leq \inf_D \frac{\int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) + \epsilon'}{\int \ell(\underline{x}|\theta) dG_n(\theta)} \quad (\text{by (2.1)}) \\ & \leq \inf_D \frac{\int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} + \epsilon. \quad (\text{by (2.6) and (2.7)}) \end{aligned}$$

Thus all $D'_n(\epsilon')$ satisfy (2.2) and it follows that $D'_n(\epsilon') \in \{D_n(\epsilon)\}$. And by construction $\epsilon' \downarrow 0$ iff $\epsilon \downarrow 0$.

$$\begin{aligned}
\lim_{\epsilon' \downarrow 0} \lim_{n \rightarrow \infty} (C'(\epsilon')) &\leq \left(\frac{3}{2}C\right) \cdot \lim_{\epsilon' \downarrow 0} \lim_{n \rightarrow \infty} \sup \left[\frac{\int L_n(\theta, D'_n(\epsilon')) \ell(\underline{x}|\theta) dF_n}{\int \ell(\underline{x}|\theta) dF_n(\theta)} - \right. \\
&\quad \left. \inf_D \frac{\int L_n(\theta, D) \ell(\underline{x}|\theta) dF_n}{\int \ell(\underline{x}|\theta) dF_n(\theta)} \right] \quad (\text{by (2.5)}) \\
&= \frac{3}{2} C \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \sup \left[\frac{\int L_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n}{\int \ell(\underline{x}|\theta) dF_n(\theta)} - \inf_D \frac{\int L_n(\theta, D) \ell(\underline{x}|\theta) dF_n}{\int \ell(\underline{x}|\theta) dF_n(\theta)} \right] \\
&= 0 \quad (\text{because } \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} C_n(\epsilon) = 0 \text{ and } D'_n(\epsilon') \in \{D_n(\epsilon)\}).
\end{aligned}$$

Similarly, we can prove the "only if" part.

- (b) If there exists $D_n(\epsilon)$, satisfying $\lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} [C_n(\epsilon)] = 0$,
we have to find $D'_n(\epsilon')$ satisfying $\lim_{\epsilon' \downarrow 0} \lim_{n \rightarrow \infty} [C'_n(\epsilon')] = 0$.

The strategy here is quite similar to part (a). Let

$$(2.8) \quad \epsilon' = \frac{3}{2} C \epsilon$$

and suppose that $D_n(\epsilon)$ is the stabilizing decision. Then

$$\frac{\int W_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} \leq \inf_D \frac{\int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} + \epsilon.$$

Then when $n > N$,

$$\begin{aligned}
\int W_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dG_n(\theta) &\leq \inf_D \int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) + \epsilon \int \ell(\underline{x}|\theta) dG_n \\
&\leq \inf_D \int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) + \epsilon'.
\end{aligned}$$

Thus $D_n(\epsilon)$ satisfies (2.1).

Let $D'_n(\epsilon') = D_n(\epsilon)$, then

$$\begin{aligned}
&\lim_{\epsilon' \downarrow 0} \lim_{n \rightarrow \infty} \sup \left[\int W_n(\theta, D'_n(\epsilon')) \ell(\underline{x}|\theta) dG_n(\theta) - \inf_D \int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) \right] \\
&\leq \frac{3}{2} C \cdot \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \sup \left[\frac{\int W_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} - \inf_D \frac{\int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta)}{\int \ell(\underline{x}|\theta) dG_n(\theta)} \right] \\
&= 0.
\end{aligned}$$

Similarly, we can prove that the "only if" part is also true. ■

From the above lemma, we can easily show the following definitions I', II', III' and IV' are equivalent to definitions I, II, III and IV respectively. Definitions I', II', III' and IV' are mathematically easier to handle.

Definition I': Let $F_n \xrightarrow{W} F_\infty$, $L_n \rightarrow L_\infty$ uniformly in θ and D , and let $D_\infty(\epsilon)$ be a decision satisfying

$$(2.9) \quad \int L_\infty(\theta, D_\infty(\epsilon)) \ell(\underline{x}|\theta) dF_\infty(\theta) \leq \inf_D \int L_\infty(\theta, D) \ell(\underline{x}|\theta) dF_\infty(\theta) + \epsilon.$$

If

$$(2.10) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left[\int L_n(\theta, D_\infty(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_n(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta) \right] = 0$$

for every choice of $L_n \rightarrow L_\infty$ uniformly in θ and D , and every $F_n \xrightarrow{W} F_\infty$ and every choice of $D_\infty(\epsilon)$ satisfying (2.9), the triple $(L_\infty, \ell(\underline{x}|\theta), F_\infty)$ is called strongly stable. There are cases in which (2.10) holds for every choice of $L_n \rightarrow L_\infty$ and $F_n \rightarrow F_\infty$, but only for some particular choice of $D_\infty(\epsilon)$. In this case, the triple $(L_\infty, \ell(\underline{x}|\theta), F_\infty)$ is called weakly stable and $D_\infty(\epsilon)$ is called a stabilizing decision. If the triple is neither strongly nor weakly stable, it is called unstable.

Definition II': Let $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$; $W_n \rightarrow L_\infty$ and $L_n \rightarrow L_\infty$ uniformly in θ and D , and let $D_n(\epsilon)$ be a decision satisfying

$$(2.11) \quad \int W_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dG_n(\theta) \leq \inf_D \int W_n(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) + \epsilon.$$

If

$$(2.12) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left[\int L_n(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_n(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta) \right] = 0$$

for every choice of $L_n \rightarrow L_\infty$, $W_n \rightarrow L_\infty$ uniformly in θ and D , and every $F_n \xrightarrow{W} F_\infty$, $G_n \xrightarrow{W} F_\infty$ and every choice of $D_n(\epsilon)$ satisfying (2.11), then the triple $(L_\infty, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is called strongly stable. If there is only some choice of $D_n(\epsilon)$ which makes (2.12) hold, then $(L_\infty, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is called weakly stable and $D_\infty(\epsilon)$ is called the stabilizing decision. If a triple is neither strongly nor weakly stable, it is called unstable.

The following two definitions, definitions III' and IV', are simpler than definitions I' and II'. We show in Chapter 3 that definitions I' and II' are equivalent to definitions III' and IV' respectively.

Definition III': Let $F_n \xrightarrow{W} F_\infty$, and $D_\infty(\epsilon)$ be a decision satisfying

$$(2.13) \quad \int L_\infty(\theta, D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) \leq \inf_D \int L_\infty(\theta, D) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + \epsilon.$$

Now let $B_n(\epsilon)$ be defined by

$$(2.14) \quad B_n(\epsilon) = \int L_\infty(\theta, D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_\infty(\theta, D) \mathcal{L}(\underline{x}|\theta) dF_n(\theta).$$

If

$$(2.15) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [B_n(\epsilon)] = 0$$

for every choice of $F_n \xrightarrow{W} F_\infty$ and every choice of $D_\infty(\epsilon)$ satisfying (2.13), the triple $(L_\infty, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is called strongly stable. There are cases in which (2.15) holds for every choice of $F_n \rightarrow F_\infty$, but only for some particular choice of $D_\infty(\epsilon)$. Then we call the triple weakly stable, and $D_\infty(\epsilon)$ is called a stabilizing decision. If a triple is neither strongly stable nor weakly stable, it is called unstable.

Definition IV': Let $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$, and $D_n(\epsilon)$ be a decision satisfying

$$(2.16) \quad \int L_\infty(\theta, D_n(\epsilon)) \lambda(\underline{x}|\theta) dG_n(\theta) \leq \inf_D \int L_\infty(\theta, D) \lambda(\underline{x}|\theta) dG_n(\theta) + \epsilon.$$

Let $C_n(\epsilon)$ be defined by

$$(2.17) \quad C_n(\epsilon) = \int L_\infty(\theta, D_n(\epsilon)) \lambda(\underline{x}|\theta) dF_n - \inf_D \int L_\infty(\theta, D) \lambda(\underline{x}|\theta) dF_n(\theta).$$

If

$$(2.18) \quad \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow +\infty} [C_n(\epsilon)] = 0$$

for every choice of $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$ and every choice of $D_n(\epsilon)$ satisfying (2.16), then the triple $(L_\infty, \lambda(\underline{x}|\theta), F_\infty)$ is called strongly stable. There are cases in which (2.18) holds for every choice of $F_n \xrightarrow{W} F_\infty$ and $G_n \xrightarrow{W} F_\infty$, but only for some particular choice of $D_n(\epsilon)$. Then we call the triple weakly stable, and $D_n(\epsilon)$ is called a stabilizing decision. If a triple is neither strongly nor weakly stable, it is called unstable.

The following theorem is a direct result of Lemma 2.1.

Theorem 2.1 Definitions I', II', III' and IV' are equivalent to definitions I, II, III and IV, respectively.

2.2 Convergence of $P_n(\theta)$

In this section, we study convergence of $P_n(\theta)$, where $P_n(\theta)$ is the posterior distribution of θ corresponding to the prior distribution $F_n(\theta)$ and the likelihood function $l(\underline{x}|\theta)$. As defined in Chapter 1, $l(\underline{x}|\theta)$ is a continuous and uniformly bounded function of θ . We also know $P_\infty(\theta)$ is the posterior distribution of θ corresponding to the prior distribution $F_\infty(\theta)$ and the likelihood function $l(\underline{x}|\theta)$.

Theorem 2.2: Under the above conditions, if $F_n \xrightarrow{W} F_\infty$ then $P_n(\theta) \xrightarrow{W} P_\infty(\theta)$.

Proof: We have to show that for every continuity point θ of P_∞ , $P_n(\theta) \rightarrow P_\infty(\theta)$.

(i) If θ is a continuity point of F_∞ , then by the Helly-Bray Theorem

$$(2.19) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_n(\theta) = \int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_\infty(\theta), \text{ and}$$

$$(2.20) \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\theta} l(\underline{x}|t) dF_n(t) = \int_{-\infty}^{\theta} l(\underline{x}|t) dF_\infty(t).$$

Thus we have

$$\lim_{n \rightarrow \infty} \frac{\int_{-\infty}^{\theta} l(\underline{x}|t) dF_n(t)}{\int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_n(\theta)} = \frac{\int_{-\infty}^{\theta} l(\underline{x}|t) dF_\infty(t)}{\int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_\infty(\theta)}$$

and $\lim_{n \rightarrow \infty} P_n(\theta) = P_\infty(\theta).$

(ii) If $\ell(\underline{x}|\theta) = 0$ and θ is not a continuous point of $F_\infty(\theta)$ then, given $\epsilon > 0$, we can find a , which is a continuous point of F_∞ , such that for all $b \in [a, \theta]$, $\ell(\underline{x}|b) < \epsilon$.

$$\int_{-\infty}^{\theta} \ell(\underline{x}|t) dF_n(t) = \int_{-\infty}^a \ell(\underline{x}|t) dF_n(t) + \int_a^{\theta} \ell(\underline{x}|t) dF_n(t)$$

$$\int_{-\infty}^{\theta} \ell(\underline{x}|t) dF_\infty(t) = \int_{-\infty}^a \ell(\underline{x}|t) dF_\infty(t) + \int_a^{\theta} \ell(\underline{x}|t) dF_\infty(t).$$

The second terms in the right hand side are less than ϵ :

$$0 \leq \int_a^{\theta} \ell(\underline{x}|t) dF_n(t) \leq \epsilon \int_a^{\theta} dF_n(t) \leq \epsilon$$

$$0 \leq \int_a^{\theta} \ell(\underline{x}|t) dF_\infty(t) \leq \epsilon \int_a^{\theta} dF_\infty(t) \leq \epsilon.$$

However, by the Helly-Bray Theorem, we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^a \ell(\underline{x}|t) dF_n(t) = \int_{-\infty}^a \ell(\underline{x}|t) dF_\infty(t).$$

Thus (2.19) and (2.20) hold in this case as well and we have

$$\lim_{n \rightarrow \infty} P_n(\theta) = P_\infty(\theta).$$

(iii) If $\ell(\underline{x}|\theta) \neq 0$ and θ is not a continuous point of $F_\infty(\theta)$,

then $P_\infty(\theta) - P_\infty(\theta-) = \frac{\ell(\underline{x}|\theta)(F_\infty(\theta) - F_\infty(\theta-))}{\int_{-\infty}^{\infty} \ell(\underline{x}|\theta) dF_\infty(\theta)}$. Thus θ is not a continuous point of $P_\infty(\theta)$.

By (i), (ii) and (iii), we conclude $P_n(\theta) \xrightarrow{W} P_\infty(\theta)$. ■

This theorem connects the results in Kadane and Chuang [1978] to the results derived from definitions I, II, III and IV. We discuss the importance of this result in Chapter 5.

Chapter 3 Relationships Among Definitions I, II, III and IV

3.1 Relationships for General Loss Function

In Section 2.1, it was shown that definitions I, II, III and IV are equivalent to definitions I', II', III' and IV' respectively. In this section, we study relationships among definitions I, II, III and IV. By using definitions I', II', III' and IV', we can see definitions I and II are equivalent to definitions III and IV respectively. Later we study the relationship between definitions III and IV.

Theorem 3.1: (a) $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is strongly (weakly) stable by definition I iff $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is strongly (weakly) stable by definition III.

(b) $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is strongly (weakly) stable by definition II iff $(L_\infty, l(\underline{x}|\theta), F_\infty)$ is strongly (weakly) stable by definition IV.

Proof: By definition $L_n(\theta, D) \rightarrow L_\infty(\theta, D)$ uniformly in θ and D means given $\epsilon > 0$, there exists N_1 such that for all $n > N_1$

$$|L_n(\theta, D) - L_\infty(\theta, D)| < \epsilon.$$

By assumption about $l(\underline{x}|\theta)$ we can find a constant B such that $|l(\underline{x}|\theta)| < B$.

Then, for all $n > N_1$

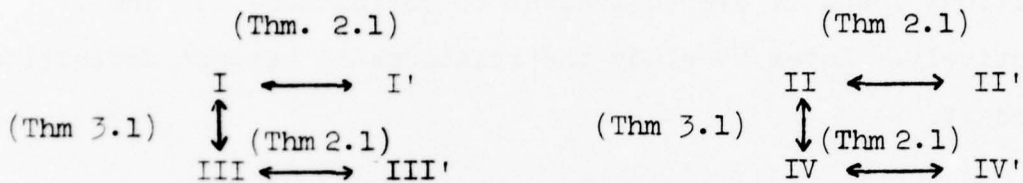
$$|L_n(\theta, D)l(\underline{x}|\theta) - L_\infty(\theta, D)l(\underline{x}|\theta)|$$

$$\leq |L_n(\theta, D) - L_\infty(\theta, D)| \cdot B$$

$$\leq B\epsilon.$$

So $L_n(\theta, D)\ell(\underline{x}|\theta) \rightarrow L_\infty(\theta, D)\ell(\underline{x}|\theta)$ uniformly in θ and D . By Theorem (1.1), (a) follows immediately. (b) can be proved in the same manner as (a). ■

From this theorem and Theorem 2.1, we have the following diagram. Where the notation " \longleftrightarrow " represents "is equivalent to."



Hence there are only two essentially different definitions to consider, which we take here to be III' and IV'. We now turn to the relationship between these two. If a triple $(L_\infty, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable by definition IV', then by letting $G_n \equiv F_\infty$ (for all n), we can see the triple is also strongly stable by definition III'. If a triple is unstable by definition III', then by the same assignment ($G_n \equiv F_\infty$), we can also see the triple is unstable by definition IV'. In Lemma 3.1 we show if a triple is weakly stable by definition III', then the triple is weakly stable by definition IV'. Here we should emphasize, as stated in our definitions, if a triple is strongly stable by a specific definition then the triple is not weakly stable by that definition. If we could show that if a triple is strongly stable by definition III' then the triple is strongly stable by definition IV', then definitions III' and IV' would be equivalent. This follows because any triple is strongly stable, weakly stable or unstable by definition IV'. If the triple is, say, strongly stable by

definition IV' then it must also be strongly stable by definition III' (for if on the contrary the triple is weakly stable (unstable) by definition III', then the triple is weakly stable (unstable) by definition IV'). Unfortunately this is not true as Example 3.1 (to come) shows.

The following lemma shows a relationship between definitions III' and IV'. Example 3.1 shows that the converse is not necessarily true.

Lemma 3.1. If a triple is weakly (not strongly) stable by definition III' then the triple is weakly (not strongly) stable by definition IV'.

Proof: The triple can not be strongly stable by definition IV'. If it is strongly stable by definition IV' then it is also strongly stable by definition III'. This is a contradiction.

Let $D_\infty(\epsilon)$ be the stabilizing decision for definition III'. Our goal is to prove $D_\infty(\epsilon)$ is also a stabilizing decision for definition IV'. By definition of stabilizing decision,

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} \left[\int L_\infty(\theta, D_\infty(\epsilon)) \lambda(\underline{x}|\theta) dG_n(\theta) - \inf_D \int L_\infty(\theta, D) \lambda(\underline{x}|\theta) dG_n(\theta) \right] = 0.$$

One thing we have to notice here is that if $\{D_\infty(\epsilon)\}$ is the set of $D_\infty(\epsilon)$ satisfying (2.1), then for any $\epsilon_1 < \epsilon$, $\{D_\infty(\epsilon_1)\} \subset \{D_\infty(\epsilon)\}$.

Thus given any $\epsilon > 0$, there exists $D_\infty(\epsilon)$ (which is a stabilizing decision satisfying)

$$\limsup_{n \rightarrow \infty} \left[\int L_\infty(\theta, D_\infty(\epsilon)) \lambda(\underline{x}|\theta) dG_n(\theta) - \inf_D \int L_\infty(\theta, D) \lambda(\underline{x}|\theta) dG_n(\theta) \right] < \frac{\epsilon}{2}.$$

From this, we can find an N , such that for all $n > N$,

$$(3.1) \quad \int L_{\infty}(\theta, D_{\infty}(\epsilon)) \ell(\underline{x}|\theta) dG_n(\theta) - \inf_D \int L_{\infty}(\theta, D) \ell(\underline{x}|\theta) dG_n(\theta) < \epsilon.$$

Comparing (3.1) with (2.16), we see $D_{\infty}(\epsilon)$ satisfies (2.16).

In order to prove weak stability by definition IV', we have to find $D_n(\epsilon)$ such that for all $F_n \xrightarrow{W} F_{\infty}$

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [\int L_{\infty}(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_{\infty}(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta)] = 0.$$

For $n > N$, we take $D_n(\epsilon) = D_{\infty}(\epsilon)$ which satisfies (2.16) and by the definition of the stabilizing decision for definition III', we have:

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [\int L_{\infty}(\theta, D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_{\infty}(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta)] \\ &= \lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [\int L_{\infty}(\theta, D_{\infty}(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int L_{\infty}(\theta, D) \ell(\underline{x}|\theta) dF_n(\theta)] \\ &= 0. \quad \blacksquare \end{aligned}$$

From now on, when we want to prove definition III' and IV' are equivalent, we only have to prove strong stability by definition III' implies strong stability by definition IV'.

The following example shows definitions III' and IV' are not equivalent for all triples, by showing a triple which is strongly stable by definition III' but only weakly stable by definition IV'. The triple works in the following way: It can be shown that the set of $D_{\infty}(\epsilon)$ for the triple is $(0, 3\epsilon)$, and the expected loss, as a function of decision D , is discontinuous at $D=0$. Thus zero is not in the set of $D_{\infty}(\epsilon)$. However, we can find $G_n \xrightarrow{W} F_{\infty}$ and zero is an ϵ -optimal decision for $(L_{\infty}, \ell(\underline{x}|\theta), G_n)$. If we

let $F_n = F_\infty$ and $D_n(\epsilon) = 0$ we can see immediately (2.18) does not hold. So the triple cannot be strongly stable by definition IV'.

Example 3.1. Suppose $L_\infty(\theta, D) = h(\theta - D)$ and

$$h(x) = \begin{cases} 2 & x \geq 1 \\ 0 & 1 > x \geq 0 \\ -x & 0 > x \geq -2 \\ 2 & -2 > x \end{cases}$$

Let $\ell(\underline{x}|\theta) \equiv 1$ and $F_\infty(\theta)$ be the cumulative distribution function of a r.v. X satisfying $P(X=-1) = \frac{1}{3}$ and $P(X=1) = \frac{2}{3}$. Then the triple $(L_\infty, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable by definition III' and only weakly stable by definition IV'.

Proof: First we show the triple is strongly stable by definition III'.

$$\int_{-\infty}^{\infty} h(\theta - D) dF_\infty(\theta) = \begin{cases} 2 & D > 3 \\ \frac{2}{3}D & 3 \geq D > 1 \\ \frac{1}{3}(1+D) & 1 \geq D > 0 \\ \frac{5}{3} + \frac{1}{3}D & 0 \geq D > -1 \\ \frac{4}{3} & -1 \geq D > -2 \\ 2 & -2 \geq D \end{cases}.$$

Since our concern is with the limiting case as $\epsilon \downarrow 0$, without loss of generality we assume $\epsilon < \frac{1}{9}$. So $D_\infty(\epsilon) \in (0, 3\epsilon)$. Note that zero is excluded from the interval.

Let $F_n(\theta)$ be any sequence of opinions converging weakly to $F_\infty(\theta)$ and let $a = D_\infty(\epsilon)/2$.

By assumptions, we know $-1-a$, $-1+a$, $1-a$ and $1+a$ are continuous points of $F_\infty(\theta)$ and $F_\infty(-1-a)=0$, $F_\infty(-1+a)=\frac{1}{3}$, $F_\infty(1-a)=\frac{1}{3}$ and $F_\infty(1+a)=1$. By definition of weak convergence we can find an N such that for all $n > N$, $F_n(-1-a) < \epsilon$, $F_n(-1+a) > \frac{1}{3} - \epsilon$, $F_n(1-a) < \frac{1}{3} + \epsilon$ and $F_n(1+a) > 1 - \epsilon$. Thus when $n > N$, we know $P(-1-a \leq \theta \leq -1+a) \geq \frac{1}{3} - 2\epsilon$ and $P(1-a \leq \theta \leq 1+a) \geq \frac{2}{3} - 2\epsilon$.

Our next goal is to show that there exists $B(\epsilon)$ such that for all $D_\infty(\epsilon)$

$$\limsup_{n \rightarrow +\infty} [\int h(\theta - D_\infty(\epsilon)) dF_n(\theta) - \inf_D \int h(\theta - D) dF_n(\theta)] < B(\epsilon)$$

and $\lim_{\epsilon \downarrow 0} B(\epsilon) = 0$.

$$\begin{aligned} \int h(\theta - D_\infty(\epsilon)) dF_n(\theta) &\leq 2 \cdot (4\epsilon) + \frac{1}{3} (1+a+D_\infty(\epsilon)) \\ &\leq 8\epsilon + \frac{1}{3} + \frac{3}{2}\epsilon \leq \frac{1}{3} + 10\epsilon. \end{aligned}$$

The reason we have first inequality is that when $n > N$, we have 4ϵ total probability about whose location we have no information. We thus use maximum loss for this set of θ . When θ is between $-1-a$ and $-1+a$, the maximum loss is $|-1-a-D_\infty(\epsilon)|$; and when θ is between $1-a$ and $1+a$ the loss is zero.

However, for any D we can see

$$\int h(\theta - D) dF_n(\theta) \geq (\frac{1}{3} - 2a)(1-2a) \geq \frac{1}{3} - 2\epsilon - \frac{2}{3}a \geq \frac{1}{3} - 3\epsilon.$$

Thus $\limsup_{n \rightarrow +\infty} [\int h(\theta - D_\infty(\epsilon)) dF_\infty(\theta) - \inf_D \int h(\theta - D) dF_n(\theta)]$

$$\leq \frac{1}{3} + 10\epsilon - \frac{1}{3} + 3\epsilon = 13\epsilon.$$

And $\lim_{\epsilon \downarrow 0} (1/3\epsilon) = 0$.

Thus the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable by definition III'.

Next we show this triple is not strongly stable by definition IV'. Let

$$G_n(\theta) = \begin{cases} 0 & \theta < -1 \\ \frac{1}{3} & -1 \leq \theta < 1 - \frac{1}{n} \\ 1 & 1 - \frac{1}{n} \leq \theta \end{cases}$$

Then we can see easily $G_n(\theta) \xrightarrow{W} F_\infty(\theta)$, and

$$\int_{-\infty}^{\infty} h(\theta-D) dG_n(\theta) = \begin{cases} 2 & D > 3 - \frac{1}{n} \\ \geq \frac{2}{3} & 3 - \frac{1}{n} \geq D > 1 - \frac{1}{n} \\ \frac{1+D}{3} & 1 - \frac{1}{n} \geq D > -\frac{1}{n} \\ \geq \frac{4}{3} & -\frac{1}{n} \geq D \end{cases}$$

Then for all $n > \frac{1}{3\epsilon}$, from the above calculation we have

$$\int_{-\infty}^{\infty} h(\theta-0) dG_n(\theta) - \inf_D \int_{-\infty}^{\infty} h(\theta-D) dG_n(\theta) = \frac{1}{3} - \frac{1 - \frac{1}{n}}{3} = \frac{1}{3n} < \epsilon.$$

Thus 0 is an ϵ -optimal decision for $(L_\infty, \mathcal{L}(\underline{x}|\theta), G_n)$.

Now let $F_n(\theta) \equiv F_\infty(\theta)$ for all n , and let $D_n(\epsilon)$ in (2.16) equal zero. Then

$$\begin{aligned} C_n(\epsilon) &= \int L_\infty(\theta, D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_n(\epsilon) - \inf_D \int L_\infty(\theta, D) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) \\ &= \int h(\theta) dF_\infty(\theta) - \inf_D \int h(\theta-D) dF_\infty(\theta) = \frac{3}{5} - \frac{1}{3} = \frac{4}{15}. \end{aligned}$$

Thus $\lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} C_n(\epsilon) \neq 0$.

And this triple is not strongly stable by definition IV'.

However, by exactly the same method as in Lemma 3.1, it can be shown that this triple is weakly stable by definition IV' and any $D_\infty(\epsilon)$ is the stabilizing decision. ■

Since in this example $\ell(\underline{x}|\theta) \equiv 1$, this example also shows that definition 1 (and 3) are not the same as definition 2 (and 4), contrary to the conjecture in Kadane and Chuang [1978].

3.2 Relationships of Definitions in Estimation or Prediction Problem

From the previous sections, we know definitions I, III, I' and III' are equivalent to each other. We also know definitions II, IV, II' and IV' are equivalent to each other. Thus, it is sufficient to study relationships between definitions III' and IV'. As we mentioned before, we choose them because they are easier to handle mathematically.

In this section, our object is to show definitions III' and IV' are equivalent for the estimation problem. By the estimation problem, we mean $L_{\infty}(\theta, D) = h(\theta - D)$, where $h(x)$ is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. Throughout this section we assume $L_{\infty}(\theta, D)$ satisfies these conditions. (Note that without the condition of continuity, Example 3.1 applies and shows that III' and IV' are not equivalent.)

We showed in Section 3.1 that in order to prove definitions III' and IV' are equivalent, it is sufficient to prove that if a triple is strongly stable by definition III' then the triple is strongly stable by definition IV'. Thus our object is to show that if a triple $(h, \mathcal{L}(\underline{x}|\theta), F_{\infty})$ is strongly stable by definition III' then the triple is strongly stable by definition IV'. We first consider those triples whose loss functions are uniformly bounded, then we consider those whose loss functions are uniformly bounded on one side and unbounded on the other side, i.e.

$\lim_{x \rightarrow -\infty} h(x) = A < \infty, \lim_{x \rightarrow +\infty} h(x) = +\infty$ or vice versa. Finally we consider

those whose loss functions are unbounded as $x \rightarrow +\infty$ and $x \rightarrow -\infty$.

We remind the reader of the following definition, previously given in Section 1.1: $h(\theta-D)$ is continuous in θ uniformly in D iff for all $\epsilon > 0$ and θ , there exists $\delta > 0$ such that for all D , $|\theta - \theta_0| < \delta$ implies

$$|h(\theta-D) - h(\theta_0-D)| < \epsilon.$$

Lemma 3.2. If $h(x)$ is uniformly bounded, then $h(\theta-D)$ is continuous in θ uniformly in D .

Proof: Since $h(x)$ is uniformly bounded, continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$, we can let

$$\lim_{x \rightarrow +\infty} h(x) = \alpha \quad \text{and} \quad \lim_{x \rightarrow -\infty} h(x) = \beta.$$

Given $\epsilon > 0$, we can find $a > 0$ and $b < 0$ such that

$$h(a) > \alpha - \frac{\epsilon}{2} \quad \text{and} \quad h(b) > \beta - \frac{\epsilon}{2}.$$

Now we have to find δ .

Since $h(x)$ is uniformly continuous in $[b, a]$, there exists $\delta_1 > 0$ such that for all $x, y \in [b, a]$, if $|x-y| < \delta_1$, then

$$|h(x) - h(y)| < \epsilon/2.$$

Let $\delta = \min(\delta_1, b-a)$.

Then for any $u, v \in \mathbb{R}$, without loss of generality we assume $u > v$. If $|u-v| < \delta$ we have to consider five cases:

- (1) $u \geq a, v \geq a$ (2) $u \geq a, a \geq v \geq b$ (3) $a \geq u \geq b, a \geq v \geq b$
 (4) $a \geq u \geq b, b \geq v$ (5) $b \geq u, b \geq v$.

For (1) $|h(u) - h(v)| \leq |\beta - (\beta - \epsilon/2)| < \epsilon$

(2) $|h(u) - h(v)| \leq h(u) - h(a) + |h(a) - h(v)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

The same inequality holds for (3), (4) and (5) also. Thus we have for any $u, v \in R$, $|u-v| < \delta$ implies

$$|h(u) - h(v)| < \epsilon.$$

Applying this result to our definition of continuous in θ uniformly in D , if $|\theta - \theta_0| < \delta$ then $|(\theta - D) - (\theta_0 - D)| = |\theta - \theta_0| < \delta$. Thus $|h(\theta - D) - h(\theta_0 - D)| < \epsilon$.

So $h(\theta - D)$ is continuous in θ uniformly in D . ■

Theorem 3.2. If $h(x)$ is uniformly bounded, then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable by definitions III' and IV'.

Proof: We show the triple is strongly stable by definition IV'. That the triple is strongly stable by definition III' follows immediately.

Let $P_\infty(\theta)$, $P_n(\theta)$ and $Q_n(\theta)$ be the posterior distributions of θ corresponding to the likelihood function $\mathcal{L}(\underline{x}|\theta)$ and the prior distributions $F_\infty(\theta)$, $F_n(\theta)$ and $G_n(\theta)$ respectively. Then by Theorem 2.2 we know $P_n(\theta) \xrightarrow{W} P_\infty(\theta)$ and $Q_n(\theta) \xrightarrow{W} P_\infty(\theta)$.

The loss function $h(\theta - D)$ is uniformly bounded and continuous in θ uniformly in D , thus by Theorem 1.2, (h, \dot{P}_∞) is strongly stable by definition 2. This implies for any $D_n(\epsilon)$ satisfying

$$\int W_n(\theta, D_n(\epsilon)) dQ_n(\theta) \leq \inf_D \int W_n(\theta, D) dQ_n(\theta) + \epsilon$$

we have

$$\lim_{\epsilon \downarrow 0} \limsup_{n \rightarrow \infty} [\int L_n(\theta, D_n(\epsilon)) dP_n(\theta) - \inf_D \int L_n(\theta, D) dP_n(\theta)] = 0$$

And this is the condition for strongly stable by definition IV'. Thus the triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable by definitions III' and IV'. ■

From Theorem 3.2 we know when h is uniformly bounded then the triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable by definitions III' and IV', and this means when the loss function is uniformly bounded, definitions III' and IV' are equivalent.

Now we consider the case that $h(x)$ is uniformly bounded on one side and unbounded on the other side. Without loss of generality we assume $\lim_{x \rightarrow +\infty} h(x) = +\infty$, $h(0) = 0$ and $\lim_{x \rightarrow -\infty} h(x) = A < \infty$. Our object is to show that a triple is strongly stable by definition III' implies the triple is strongly stable by definition IV'. The following lemma eliminates some triples from consideration.

The proof in the following lemma is important in the sense we use the same procedure to prove a triple is unstable or weakly stable in this and the following chapters.

Lemma 3.3. If $\lim_{x \rightarrow +\infty} h(x) = +\infty$, $h(0) = 0$, $\lim_{x \rightarrow -\infty} h(x) = A < \infty$, and if for any $\epsilon > 0$, there exists $D_\infty(\epsilon)$ such that $h(\theta - D_\infty(\epsilon))\ell(\underline{x}|\theta)$ as a function of θ is not bounded above, then the triple is not strongly stable by Definition III'.

Proof: We prove the lemma by finding $D_\infty(\epsilon)$ and F_n such that (2.15) does not hold. Since $\lim_{D \rightarrow -\infty} \int h(\theta - D)\ell(\underline{x}|\theta)dF_\infty(\theta) = +\infty$, we know $D_\infty(\epsilon)$ is uniformly bounded below.

One important fact about our loss function and likelihood function is that when $h(\theta-D)\ell(\underline{x}|\theta)$ is not bounded above then for all $d < D$, $h(\theta-d)\ell(\underline{x}|\theta)$ is also not bounded above. This is true because $h(\theta-D)\ell(\underline{x}|\theta)$ goes to infinity only when θ goes to $+\infty$. However, when $\theta > D$ we know $h(\theta-D) \leq h(\theta-d)$, and thus $h(\theta-D)\ell(\underline{x}|\theta) \leq h(\theta-d)\ell(\underline{x}|\theta)$. So when $h(\theta-D)\ell(\underline{x}|\theta)$ is unbounded implies $h(\theta-d)\ell(\underline{x}|\theta)$ is also unbounded.

Let $\epsilon_1, \epsilon_2, \dots$ be a sequence such that $\epsilon_1 > 0$ and $\epsilon_1 \downarrow 0$. Our object is to show there exists $F_n(\theta)$ such that for any $\epsilon > 0$ there exists $D_\infty(\epsilon)$ such that (2.15) does not hold.

By our assumptions, for any ϵ_1 , we can find $D_\infty(\epsilon_1)$ such that $h(\theta-D_\infty(\epsilon_1))\ell(\underline{x}|\theta)$ is not uniformly bounded.

Let $D_1 = \max\{D_\infty(\epsilon_1), D_\infty(\epsilon_2), \dots, D_\infty(\epsilon_1)\}$.

Thus $h(\theta-D_1)\ell(\underline{x}|\theta)$ is not uniformly bounded above. And by the discussion at the beginning of the proof, $\forall k \leq 1$ and $\theta > D_1$

$$h(\theta-D_1)\ell(\underline{x}|\theta) \leq h(\theta-D_\infty(\epsilon_k))\ell(\underline{x}|\theta).$$

Next we construct $F_n(\theta)$ such that $F_n(\theta)$ and $D_\infty(\epsilon_1)$ makes (2.15) fail.

We can find θ_1 such that $\theta_1 > D_1$ and $h(\theta_1-D_1)\ell(\underline{x}|\theta_1) > i^2$, because $h(\theta-D_1)\ell(\underline{x}|\theta)$ is not uniformly bounded above.

Now let $F_n(\theta) = (1 - \frac{1}{n})F_\infty(\theta) + \frac{1}{n}J_n(\theta)$ where $J_n(\theta)$ is defined as

$$J_n(\theta) = \begin{cases} 0 & \theta < \theta_n \\ 1 & \theta \geq \theta_n \end{cases}.$$

Then for any i , consider $n > 1$

$$\begin{aligned}
 & \int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(\underline{x}|\theta) dF_n(\theta) \\
 & \geq (1 - \frac{1}{n}) \left(\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_\infty(\theta) - \int h(\theta - \theta_n) \ell(\underline{x}|\theta) dF_\infty(\theta) \right) \\
 & \quad + \frac{1}{n} h(\theta_n - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta_n) \\
 & \geq \frac{1}{n} \cdot n^2 - \int h(\theta - \theta_n) \ell(\underline{x}|\theta) dF_\infty(\theta).
 \end{aligned}$$

However, the last term

$$\begin{aligned}
 & \int h(\theta - \theta_n) \ell(\underline{x}|\theta) dF_\infty(\theta) \\
 & = \left[\int_{-\infty}^{\theta_n} h(\theta - \theta_n) \ell(\underline{x}|\theta) + \int_{\theta_n}^{\infty} h(\theta - \theta_n) \ell(\underline{x}|\theta) \right] dF_\infty(\theta) \\
 & \leq \int_{-\infty}^{\theta_n} A \cdot \ell(\underline{x}|\theta) dF_\infty(\theta) + \int_{\theta_n}^{\infty} h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_\infty(\theta) \\
 & \leq A \cdot B + \int_{-\infty}^{\infty} h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_\infty(\theta).
 \end{aligned}$$

where $\ell(\underline{x}|\theta) \leq B$. Thus this term is bounded by a value which is independent of n .

And for all $D_\infty(\epsilon_1)$

$$\begin{aligned}
 & \lim_{n \rightarrow +\infty} \sup \left[\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(\underline{x}|\theta) dF_n(\theta) \right] \\
 & \geq \lim_{n \rightarrow \infty} \sup \left[\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - \theta_n) \ell(\underline{x}|\theta) dF_n(\theta) \right]
 \end{aligned}$$

$\rightarrow +\infty$.

Thus the triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is not strongly stable by definition III'. ■

The following lemma, an application of the Helly-Bray Theorem (Loeve 1963, page 181), is very **useful** throughout the whole thesis.

Lemma 3.4: Suppose $h(x)$ is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$, and let a, b, c, d be any numbers satisfying $c \leq a < b \leq d$. Then for any $F_n \xrightarrow{W} F_\infty$ and $\epsilon > 0$ we can find an N such that $\forall n > N$, and $\forall d_1, d_2 \in [a, b]$.

$$(3.2) \dots \dots \int_c^d (h(\theta - d_1) - h(\theta - d_2)) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) \\ \leq \int_c^d (h(\theta - d_1) - h(\theta - d_2)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + \epsilon$$

Proof. Let $\mathcal{L}(\underline{x}|\theta) < B$.

We know for any two specific D_1 and D_2 in $[a, b]$, there exists N , which is a function of D_1 and D_2 , such that for all $n > N$,

$$\int_c^d (h(\theta - D_1) - h(\theta - D_2)) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) < \int_c^d (h(\theta - D_1) - h(\theta - D_2)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + \epsilon.$$

How to make the N in the above independent of D_1 and D_2 is our goal.

We know $h(x)$ is uniformly continuous in $[c-b, d-a]$, so we can find $\delta > 0$ such that if $|x-y| < \delta$, $x, y \in [c-b, d-a]$ then

$$|h(x) - h(y)| < \epsilon/8B.$$

And $[a, b]$ is compact, so we can find a finite open covering of $[a, b]$, $\{(e_i, f_i), i=1, 2, \dots, k\}$ such that for all i , $f_i - e_i < \delta$.

Let $t_i \in (e_i, f_i)$ and $t_i \in [a, b]$.

The following inequality shows that if two decisions are very close, then for any distribution F , $\int_c^d (h(\theta - d_1) - h(\theta - d_2)) \cdot \mathcal{L}(\underline{x}|\theta) dF(\theta)$ is very small. Because of this inequality, we can

reduce the total number of decisions we have to consider for (3.2).

Let $g \in (e_1, f_1)$ and for any F ,

$$|(\theta - g) - (\theta - t_1)| = |g - t_1| < \delta \quad \text{and for all}$$

$$\theta \in [c, d], \quad |h(\theta - g) - h(\theta - t_1)| < \epsilon/8B.$$

$$\text{So } \int_c^d |h(\theta - g) - h(\theta - t_1)| \ell(\underline{x}|\theta) dF(\theta) < \int_c^d (\epsilon/8B) \cdot \ell(\underline{x}|\theta) dF(\theta)$$

$$(3.3) \quad \dots \quad \epsilon/8 < \int_c^d (h(\theta - g) - h(\theta - t_1)) \ell(\underline{x}|\theta) dF(\theta) < \epsilon/8$$

Now by the Helly-Bray Theorem, we can find N_{1j} such that for all $n > N_{1j}$

$$(3.4) \quad \dots \quad \int_c^d (h(\theta - t_1) - h(\theta - t_j)) \ell(\underline{x}|\theta) dF_n(\theta) \\ < \int_c^d (h(\theta - t_1) - h(\theta - t_j)) \ell(\underline{x}|\theta) dF_\infty(\theta) + \frac{\epsilon}{2}.$$

Combining (3.3) and (3.4) together, for all $n > \max\{N_{1j}, 1=1,2, \dots, k, j=1,2, \dots, k\}$, and all d_1 and $d_2 \in [a, b]$, we have the following inequalities. Without loss of generality, we assume $d_1 \in (e_1, f_1)$ and $d_2 \in (e_j, f_j)$.

$$\begin{aligned} & \int_c^d h(\theta - d_1) \ell(\underline{x}|\theta) dF_n(\theta) - \int_c^d h(\theta - d_2) \ell(\underline{x}|\theta) dF_n(\theta) \\ & \leq \int_c^d h(\theta - t_1) \ell(\underline{x}|\theta) dF_n(\theta) + \frac{\epsilon}{8} - \int_c^d h(\theta - t_j) \ell(\underline{x}|\theta) dF_n(\theta) + \epsilon/8 \quad (\text{by (3.3)}) \\ & \leq \int_c^d (h(\theta - t_1) - h(\theta - t_j)) \ell(\underline{x}|\theta) dF_\infty(\theta) + \frac{3}{4} \epsilon \quad (\text{by (3.4)}) \\ & \leq \int_c^d (h(\theta - d_1) - h(\theta - d_2)) \ell(\underline{x}|\theta) dF_\infty(\theta) + \epsilon \quad (\text{by 3.3}). \quad \blacksquare \end{aligned}$$

After Lemma 3.3, for those triples whose loss function is uniformly bounded on one side and unbounded on the other side, we

have to consider only the case given in the following lemma.

Lemma 3.5. Suppose there exists $\epsilon_0 > 0$ such that for any $D_\infty(\epsilon_0)$, $h(\theta - D_\infty(\epsilon_0))\ell(\underline{x}|\theta)$ as a function of θ is uniformly bounded. If the triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable by definition III' then it is strongly stable by definition IV'.

Proof: There are two mutually exclusive cases which we would like to consider separately:

(a) there exists $\epsilon > 0$ such that $D_\infty(\epsilon)$ is uniformly bounded

(b) for all $\epsilon > 0$ and $c > 0$, there exists $D_\infty(\epsilon) > c$.

Before we start to prove these two cases, we need some preliminary results.

By $\lim_{x \rightarrow +\infty} h(x) = +\infty$, we know $\lim_{D \rightarrow -\infty} \int h(\theta - D)\ell(\underline{x}|\theta)dF_\infty(\theta) = +\infty$,

thus $D_\infty(\epsilon)$ is uniformly bounded below.

Let H be any distribution function and $g(D) = \int h(\theta - D)dH(\theta)$. We next show $\lim_{D \rightarrow \infty} g(D) = A$ and $\lim_{D \rightarrow -\infty} g(D) = +\infty$. These can be seen easily, since for any θ , $\lim_{D \rightarrow \infty} h(\theta - D) = A$ and $\lim_{D \rightarrow -\infty} h(\theta - D) = +\infty$.

However, the following is the rigorous proof.

$g(0) = \int h(\theta)dH(\theta)$, and given $\epsilon > 0$, we can find $d > 0$ such that $\int_{-\infty}^{-d} h(\theta)dH(\theta) < \epsilon$.

Let $H(b) > 1 - \epsilon$ and D_0 be so large that $D_0 > d$ and $h(b - D_0) > A - \epsilon$.

Then when $D > D_0$,

$$\int h(\theta - D)dH(\theta) = \int_{-\infty}^D h(\theta - D)dH(\theta) + \int_D^{\infty} h(\theta - D)dH(\theta) \leq A + \epsilon$$

$$\int h(\theta - D)dH(\theta) \geq \int_{-\infty}^b h(\theta - D)dH(\theta) \geq (A - \epsilon)(1 - \epsilon) \geq A - (1 + A)\epsilon.$$

Thus $\lim_{D \rightarrow +\infty} g(D) = A$.

It is trivial to show $\lim_{D \rightarrow -\infty} g(D) = +\infty$.

(a) for this part, we first show that when n is large $D_n(\epsilon)$ is uniformly bounded.

Suppose on the contrary, we can find a $G_n(\theta) \xrightarrow{W} F_\infty(\theta)$ such that $D_n(\epsilon) \rightarrow +\infty$. Then

$$\begin{aligned} & \int h(\theta - D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) - \inf_D \int h(\theta - D) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) \\ & \geq \int h(\theta - D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) - \int h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) \\ & = (\int \mathcal{L}(\underline{x}|\theta) dG_n(\theta)) (\int h(\theta - D_n(\epsilon)) dQ_n(\theta) - \int h(\theta - D_\infty(\epsilon)) dQ_n(\theta)), \end{aligned}$$

where $Q_n(\theta)$ is defined in definition II.

We know $D_n(\epsilon) \rightarrow +\infty$ (as we assumed) and $Q_n(\theta) \xrightarrow{W} P_\infty(\theta)$, thus we can find an N_0 such that for all $n > N_0$, $\int h(\theta - D_n(\epsilon)) dQ_n(\theta) > A - \epsilon$, this follows directly from the preliminary results at the beginning of this lemma.

We also know $h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta)$ is uniformly bounded, thus by the Helly-Bray Theorem, we can find N_1 such that for all $n > N_1$,

$$\frac{\int h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dG_n(\theta)}{\int \mathcal{L}(\underline{x}|\theta) dG_n(\theta)} < \frac{\int h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta)}{\int \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta)} + \epsilon.$$

Thus

$$\begin{aligned} & [\int h(\theta - D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) - \inf_D \int h(\theta - D) \mathcal{L}(\underline{x}|\theta) dG_n(\theta)] \\ & \geq (\int \mathcal{L}(\underline{x}|\theta) dG_n(\theta)) (A - \epsilon - \epsilon - \int h(\theta - D_\infty(\epsilon)) dP_\infty(\theta)). \end{aligned}$$

However, $\lim_{\epsilon \downarrow 0} (A - \epsilon - \epsilon - \int h(\theta - D_\infty(\epsilon)) dP_\infty(\theta)) > 0$ since

$A = \lim_{D \rightarrow +\infty} \int h(\theta - D) dP_\infty$, and $D_\infty(\epsilon)$ is uniformly bounded.

This contradicts our definition of $D_n(\epsilon)$.

Now we can assume there exists $[a, b]$ and N such that for all $n > N$, $D_n(\epsilon) \in [a, b]$.

Let c and d be continuous points of $F_\infty(\theta)$ and satisfying $c < a$, $d > b$ and the following inequalities.

$$(3.5) \quad \dots \dots \int_{-\infty}^c (h(\theta - b) - h(\theta - a)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) < \epsilon/2$$

$$(3.6) \quad \dots \dots \int_d^{\infty} (h(\theta - a) - h(\theta - b)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) < \epsilon/2$$

and $F_\infty(c) < \epsilon/2$, $F_\infty(d) > 1 - \epsilon/2$.

Our strategy in this is to show that we can find a constant r such that for all $D_n(\epsilon)$,

$$\int L_\infty(\theta, D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) \leq \inf_D \int L_\infty(\theta, D) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + r\epsilon.$$

We first consider $\theta \in [c, d]$, later we take care of $\theta \in (-\infty, c) \cup (d, \infty)$.

We can find N_2 such that $n > N_2$, $G_n(c) < \epsilon$ and $G_n(d) > 1 - \epsilon$. And $h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta)$ is uniformly bounded implies there exists B_0 such that $|h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta)| < B_0$.

Now by Lemma 3.4, there exists N_3 such that for all $n > N_3$ and for all $d_1, d_2 \in [a, b]$,

$$\int_c^d (h(\theta - d_1) - h(\theta - d_2)) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) \geq \int_c^d (h(\theta - d_1) - h(\theta - d_2)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + \epsilon$$

In particular, let $d_2 = D_\infty(\epsilon)$, we have

$$(3.7) \quad \int_c^d (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) \geq \\ \int_c^d (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - \epsilon.$$

Combining (3.7) and $-h(\theta-D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) > -B_0$ we have,

$$(3.8) \quad \int (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) \geq \\ \int_c^d (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - (2B_0+1)\epsilon.$$

By (3.5) we know $\int_{-\infty}^c (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) < \epsilon/2$, and

by (3.6) we have $\int_d^\infty (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) < \epsilon/2$.

Combine (3.8) and the above two inequalities,

$$\int_{-\infty}^\infty (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dG_n(\theta) + \epsilon \geq \\ \int_{-\infty}^\infty (h(\theta-d_1) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - (2B_0+1)\epsilon.$$

Now let $d_1 = D_n(\epsilon)$ and by definition of $D_n(\epsilon)$,

$$\epsilon + \epsilon \geq \int_{-\infty}^\infty (h(\theta-D_n(\epsilon)) - h(\theta-D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - (2B_0+1)\epsilon.$$

So $(2B_0+4)\epsilon \geq \int_{-\infty}^\infty h(\theta-D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - \inf_D \int h(\theta-D) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta).$

This shows all $D_n(\epsilon)$ are $(2B_0+4)\epsilon$ -optimal decision for the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$.

By assumption, the triple is strongly stable by definition III', and let $\epsilon' = (2B_0 + 4)\epsilon$, then for every $D_n(\epsilon)$ we have

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \sup \left[\int h(\theta - D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(\underline{x}|\theta) dF_n(\theta) \right] \\ &= \lim_{\epsilon' \downarrow 0} \lim_{n \rightarrow \infty} \left(\int h(\theta - D_\infty(\epsilon')) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(\underline{x}|\theta) dF_n(\theta) \right) \\ &= 0. \end{aligned}$$

Thus the triple is strongly stable by definition IV'.

(b) We showed at the beginning of the proof that $\lim_{D \rightarrow +\infty} g(D) = A$, so under the assumptions of (b) we know for all $\epsilon > 0$ there exists $c(\epsilon)$ such that for all $D > c(\epsilon)$ implies D satisfies (2.9). We also showed $D_\infty(\epsilon)$ is uniformly bounded below.

By $\lim_{D \rightarrow -\infty} g(D) = +\infty$, we can see that $D_n(\epsilon)$ is uniformly bounded below. Let this lower bound be a and also let $b = c(\epsilon)$.

Now $D_n(\epsilon)$ may be in $[a, b]$ or $[b, \infty)$. We discussed the first case in (a). When $D_n(\epsilon) \in [b, \infty)$, by definition it satisfies (2.9). Thus for every $D_n(\epsilon)$,

$$\lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \left(\int h(\theta - D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(\underline{x}|\theta) dF_n(\theta) \right) = 0.$$

And the triple is strongly stable by definition IV'. ■

We thus have shown that if $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $\lim_{x \rightarrow -\infty} h(x) = A < \infty$ then definitions III' and IV' are equivalent. Similarly it can be shown that if $\lim_{x \rightarrow +\infty} h(x) = B < \infty$ and $\lim_{x \rightarrow -\infty} h(x) = \infty$ then definitions III' and IV' are equivalent. This concludes our study of relationships of definitions III' and IV' for those triples whose loss functions

are uniformly bounded on one side and unbounded on the other side.

Now the only case that we have not discussed is the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ such that $\lim_{x \rightarrow +\infty} h(x) = +\infty$. We divide them into two groups. In the first group, the triples satisfy the condition that for any $\epsilon > 0$ there exists $D_\infty(\epsilon)$ and D_0 such that $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_0))\mathcal{L}(\underline{x}|\theta)$ as a function of θ is not uniformly bounded above. In the second group, the triples satisfy the condition that there exists $\epsilon_0 > 0$ such that for all $D_\infty(\epsilon_0)$ and D_0 , $(h(\theta - D_\infty(\epsilon_0)) - h(\theta - D_0))\mathcal{L}(\underline{x}|\theta)$ is uniformly bounded above. We first show those triples in the first group are not strongly stable by definition III'. The procedure in the following lemma is quite similar to Lemma 3.3.

Lemma 3.6: If for any $\epsilon > 0$, there exists $D_\infty(\epsilon)$ and D_0 such that $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_0))\mathcal{L}(\underline{x}|\theta)$, as a function of θ , is not uniformly bounded above, then $(h, \mathcal{L}(\underline{x}|\theta)F_\infty)$ is not strongly stable by definition III'.

Proof: Let $\epsilon'_1, \epsilon'_2, \dots$ be a sequence of positive numbers such that $\epsilon'_1 \downarrow 0$.

We can find $D_\infty(\epsilon'_1)$ and D_{01} such that $(h(\theta - D_\infty(\epsilon'_1)) - h(\theta - D_{01}))\mathcal{L}(\underline{x}|\theta)$ as a function of θ is not uniformly bounded above. We can see that $D_\infty(\epsilon'_1) \neq D_{01}$.

Let $A = \{i | D_\infty(\epsilon'_1) < D_{01}\}$, $B = \{i | D_\infty(\epsilon'_1) > D_{01}\}$. Then for any $i \in I$, where I is the set of all positive integers, we have $i \in A$ or $i \in B$. Thus the total number of elements in A plus the total number of elements in B is infinity. Without loss of generality we may assume the total number of elements in A is infinity.

Then we have a subsequence of $\epsilon'_1, \epsilon'_2, \dots$, we call it $\epsilon_1, \epsilon_2, \dots$ such that $\epsilon_1 > 0$ and $\epsilon_1 \downarrow 0$ and we can find $D_\infty(\epsilon_1)$ and D_{11} such that $D_\infty(\epsilon_1) < D_{11}$ and $(h(\theta - D_\infty(\epsilon_1)) - h(\theta - D_{11}))\ell(\underline{x}|\theta)$ is not uniformly bounded above.

Our object is to find $F_n(\theta) \xrightarrow{W} F_\infty(\theta)$ such that when n is large, $D_\infty(\epsilon_1)$ has very high loss compared with minimum loss for $(h, \ell(\underline{x}|\theta), F_n)$.

When $(h(\theta - D_\infty(\epsilon_1)) - h(\theta - D_{11}))\ell(\underline{x}|\theta)$ is not uniformly bounded above and $D_\infty(\epsilon_1) < D_{11}$, then for all $D \leq D_\infty(\epsilon_1)$ and $d \geq D_{11}$, $(h(\theta - D) - h(\theta - d))\ell(\underline{x}|\theta)$, as a function of θ is also not uniformly bounded above. From this we have the following definitions and results.

$$\text{Let } D_1 = \max\{D_\infty(\epsilon_1), D_\infty(\epsilon_2), \dots, D_\infty(\epsilon_i)\}$$

$$d_1 = \max\{D_{11}, D_{12}, \dots, D_{1i}\}.$$

Then $(h(\theta - D_1) - h(\theta - d_1))\ell(\underline{x}|\theta)$ is not uniformly bounded above. And we can find $\theta_1 > d_1 \geq D_1$ such that

$$(3.9) \quad \dots \dots (h(\theta_1 - D_1) - h(\theta_1 - d_1))\ell(\underline{x}|\theta_1) > 1^2 + 1 \int (h(\theta - d_1) - h(\theta - D_\infty(\epsilon_1))) \cdot \ell(\underline{x}|\theta) dF_\infty(\theta).$$

Then when $n > 1$

$$(3.10) \quad \dots \dots (h(\theta_n - D_\infty(\epsilon_1)) - h(\theta_n - d_n))\ell(\underline{x}|\theta_n) \geq (h(\theta_n - D_n) - h(\theta_n - d_n))\ell(\underline{x}|\theta_n) \geq n^2 + n \cdot \int (h(\theta - d_n) - h(\theta - D_\infty(\epsilon_1)))\ell(\underline{x}|\theta) dF_\infty(\theta).$$

Now, we construct $F_n(\theta)$ by $F_n(\theta) = (1 - \frac{1}{n})F_\infty(\theta) + \frac{1}{n} J_n(\theta)$,
 where $J_n(\theta)$ is defined as

$$J_n(\theta) = \begin{cases} 0 & \theta < \theta_n \\ 1 & \theta \geq \theta_n \end{cases}.$$

We next show $F_n(\theta)$ satisfies our goal.

For any ϵ , let $\epsilon_1 < \epsilon$, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup \left[\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(\underline{x}|\theta) dF_n(\theta) \right] \\ & \geq \lim_{n \rightarrow \infty} \sup \left[\left(\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - d_n) \ell(\underline{x}|\theta) dF_n(\theta) \right) \right] \\ & = \lim_{n \rightarrow \infty} \sup \left[\left(1 - \frac{1}{n} \right) \left(\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_\infty(\theta) - \int h(\theta - d_n) \ell(\underline{x}|\theta) dF_\infty(\theta) \right) \right. \\ & \quad \left. + \frac{1}{n} (h(\theta_n - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta_n) - \frac{1}{n} h(\theta_n - d_n) \ell(\underline{x}|\theta_n)) \right] \\ & \geq \lim_{n \rightarrow \infty} \sup \left(\left(1 - \frac{1}{n} \right) \left(\int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_\infty(\theta) - \int h(\theta - d_n) \ell(\underline{x}|\theta) dF_\infty(\theta) \right) + \right. \\ & \quad \left. n + \int (h(\theta - d_n) - h(\theta - D_\infty(\epsilon_1))) \ell(\underline{x}|\theta) dF_\infty(\theta) \right) \quad (\text{by (3.10)}) \\ & \geq \lim_{n \rightarrow \infty} \sup \left(n + \frac{1}{n} \int h(\theta - d_n) \ell(\underline{x}|\theta) dF_\infty(\theta) - \epsilon_1 - \frac{1}{n} \int h(\theta - D_\infty(\epsilon_1)) \ell(\underline{x}|\theta) dF_\infty(\theta) \right) \\ & = +\infty. \end{aligned}$$

Thus the triple is not strongly stable by definition III'. ■

As stated at the beginning of this section, our object in this section is to show if a triple is strongly stable by definition III' then the triple is strongly stable by definition IV', because if this is true then definitions III' and IV' are equivalent. The

above and the following lemmas give us information about what kinds of triples we do not have to consider. In the following we assume $D_n(\epsilon)$ satisfies (2.16).

Lemma 3.7: If $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and there exists $F_n \xrightarrow{W} F_\infty$ and $D_n(\epsilon)$

such that $D_n(\epsilon) \rightarrow +\infty$ or $-\infty$, then $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable by definition III'.

Proof: Let $D_m(\epsilon) = \inf\{D_\infty(\epsilon)\}$, where $\{D_\infty(\epsilon)\}$ is the set of all decisions satisfying (2.9). We can see $D_m(\epsilon)$ is uniformly bounded and when $\epsilon_1 \leq \epsilon_2$ then $D_m(\epsilon_1) \geq D_m(\epsilon_2)$.

Without loss of generality, we assume $D_n(\epsilon) \rightarrow +\infty$ as $n \rightarrow +\infty$.

Since $\lim_{x \rightarrow +\infty} h(x) = +\infty$, we can find E_n satisfying $D_m(\epsilon) < E_n < D_n(\epsilon)$

and

$$(3.11) \quad h(E_n - D_m(\epsilon)) = h(E_n - D_n(\epsilon)),$$

using the continuity of h .

By assumption, we know $D_n(\epsilon) \rightarrow +\infty$ as $n \rightarrow \infty$, thus we can see $\lim_{n \rightarrow \infty} E_n = +\infty$.

In order to prove a triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable by definition III', we have to find a sequence $H_n \xrightarrow{W} F_\infty$ such that for all $D_\infty(\epsilon)$, (2.15) does not hold. Let

$$H_n(\theta) = \begin{cases} \frac{2F_n(E_n) - 1}{F_n(E_n)} \cdot F_n(\theta) & \theta \leq E_n \\ 2F_n(\theta) - 1 & \theta > E_n \end{cases}$$

Then we can see that $H_n(\theta) \xrightarrow{W} F_\infty(\theta)$.

For all $D_\infty(\epsilon)$, the following inequalities are straightforward.

$$\begin{aligned}
 & \int h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dH_n(\theta) - \inf_D \int h(\theta - D) \mathcal{L}(\underline{x}|\theta) dH_n(\theta) \\
 & \geq \int h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dH_n(\theta) - \int h(\theta - D_n(\epsilon)) \mathcal{L}(\underline{x}|\theta) dH_n(\theta) \\
 & = \int_{-\infty}^{E_n} (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) \mathcal{L}(\underline{x}|\theta) \frac{2F_n(E_n) - 1}{F_n(E_n)} dF_n(\theta) + \\
 & \quad \int_{E_n}^{\infty} (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) \mathcal{L}(\underline{x}|\theta) \cdot 2dF_n(\theta) \\
 & \geq -2\epsilon + \frac{1}{F_n(E_n)} \int_{-\infty}^{E_n} (h(\theta - D_n(\epsilon)) - h(\theta - D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_n(\theta).
 \end{aligned}$$

Now looking at the last term, we know $F_n(E_n) \rightarrow 1$ as $n \rightarrow +\infty$. And for all $\theta \in (-\infty, E_n)$, $h(\theta - D_n(\epsilon)) - h(\theta - D_\infty(\epsilon)) \geq 0$. This is true, since when $\theta < D_\infty(\epsilon)$, $\theta - D_n(\epsilon) < \theta - D_\infty(\epsilon) < 0$ thus $h(\theta - D_n(\epsilon)) > h(\theta - D_\infty(\epsilon))$; and when $D_\infty(\epsilon) < \theta < E_n$, $h(\theta - D_\infty(\epsilon)) < h(\theta - D_m(\epsilon)) < h(E_n - D_n(\epsilon)) = h(E_n - D_n(\epsilon)) < h(\theta - D_n(\epsilon))$. Let a, b be continuous points of $P_\infty(\theta)$ satisfying $P_\infty(a) < \frac{1}{3}$ and $P_\infty(b) > \frac{2}{3}$.

$$\begin{aligned}
 & \text{Thus} \\
 & \lim_{n \rightarrow \infty} \left(-2\epsilon + \frac{1}{F_n(E_n)} \int_{-\infty}^{E_n} (h(\theta - D_n(\epsilon)) - h(\theta - D_\infty(\epsilon))) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) \right) \\
 & \geq \lim_{n \rightarrow \infty} \left[-2\epsilon + \left(\int_a^b (h(\theta - D_n(\epsilon)) - h(\theta - D_\infty(\epsilon))) dP_n \right) \cdot \left(\int \mathcal{L}(\underline{x}|\theta) dF_n \right) \right] \\
 & \rightarrow \infty.
 \end{aligned}$$

This is true because when n goes to ∞ , all the terms are finite except the term $\int_a^b h(\theta - D_n(\epsilon)) dP_n(\theta)$, which goes to infinity. Thus the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable by definition III'. ■

Now we are ready to see the final lemma in this section.

Lemma 3.8. If $\lim_{x \rightarrow +\infty} h(x) = +\infty$, and the triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable by definition III' then the triple is also strongly stable by definition IV'.

Proof: By Lemma 3.6, we can find $\epsilon_0 > 0$ such that $\forall D_\infty(\epsilon_0)$ and D $(h(\theta - D_\infty(\epsilon_0)) - h(\theta - D))\ell(\underline{x}|\theta)$ is uniformly bounded above.

To free us from considering end points, let $\epsilon_1 = \epsilon_0/2$ and let $D_0 = \inf\{D_\infty(\epsilon_1)\}$, $D_1 = \sup\{D_\infty(\epsilon_1)\}$ where $\{D_\infty(\epsilon_1)\}$ is the set of all decisions satisfying (2.9) except ϵ is changed into ϵ_1 . By $\lim_{x \rightarrow +\infty} h(x) = +\infty$, we know both D_0 and D_1 are finite and $D_0 < D_1$.

Furthermore, we have the following results:

for any $D \in [D_0, D_1]$ and any d
 $(h(\theta - D) - h(\theta - d))\ell(\underline{x}|\theta)$ is uniformly bounded above, and
 for any $D_2, D_3 \in [D_0, D_1]$,
 $(h(\theta - D_2) - h(\theta - D_3))\ell(\underline{x}|\theta)$ is uniformly bounded (both above and below).

By Lemma 3.7 we know there exists $[a, b], N$, such that $n > N$, $D_n(\epsilon) \in [a, b]$. Without loss of generality assume $a < D_0$ and $b > D_1$.

Our object now is to show when ϵ is small enough, we can find N_1 such that for all $n > N_1$, $D_n(\epsilon) \in [D_0, D_1]$.

We can find c, d such that $c < a < b < d$ and, c, d continuous points of F_∞ ,

$$\int_{-\infty}^c (h(\theta - b) - h(\theta - a))\ell(\underline{x}|\theta) dF_\infty(\theta) < \frac{\epsilon}{2}$$

$$\int_d^{\infty} h(\theta - a) - h(\theta - b)\ell(\underline{x}|\theta) dF_\infty(\theta) < \frac{\epsilon}{2} \quad \text{and}$$

$$F_\infty(c) < \epsilon, \quad F_\infty(d) > 1 - \epsilon.$$

By Lemma 3.4 we have:

there exists N_2 , such that for all $n > N_2$ and for all $u, v \in [a, b]$,

$$\int_c^d (h(\theta-u) - h(\theta-v)) \ell(\underline{x}|\theta) dG_n(\theta) \geq \int_c^d (h(\theta-u) - h(\theta-v)) \ell(\underline{x}|\theta) dF_\infty(\theta) - \epsilon.$$

Let $v = D_\infty(\epsilon)$, then we can find B such that

$$(h(\theta-u) - h(\theta-D_\infty(\epsilon))) \ell(\underline{x}|\theta) \geq -B.$$

Now for any $u \in [a, b]$,

$$\begin{aligned} & \int_{-\infty}^{\infty} (h(\theta-u) - h(\theta-D_\infty(\epsilon))) \ell(\underline{x}|\theta) dG_n(\theta) \\ &= \int_{-\infty}^c + \int_c^d + \int_d^{\infty} (h(\theta-u) - h(\theta-D_\infty(\epsilon))) \ell(\underline{x}|\theta) dG_n(\theta) \\ &\geq (-B)\epsilon + \int_c^d (h(\theta-u) - h(\theta-D_\infty(\epsilon))) \ell(\underline{x}|\theta) dF_\infty(\theta) - \epsilon \\ &\geq (-B-1)\epsilon + \int_{-\infty}^{\infty} (h(\theta-u) - h(\theta-D_\infty(\epsilon))) \ell(\underline{x}|\theta) dF_\infty(\theta) - \epsilon. \end{aligned}$$

In particular, let $u = D_n(\epsilon)$, we have

$$(B+3)\epsilon \geq \int_{-\infty}^{\infty} (h(\theta-D_n(\epsilon)) - h(\theta-D_\infty(\epsilon))) \ell(\underline{x}|\theta) dF_\infty(\theta). \quad \text{Thus}$$

$$(B+4)\epsilon \geq \int_{-\infty}^{\infty} h(\theta-D_n(\epsilon)) \ell(\underline{x}|\theta) dF_\infty(\theta) - \inf_D \int h(\theta-D) \ell(\underline{x}|\theta) dF_\infty(\theta).$$

Thus when n is large, all $D_n(\epsilon)$ are in $[D_0, D_1]$ and $D_n(\epsilon)$ is a $(4+B)\epsilon$ -optimal decision of $(h, \ell(\underline{x}|\theta), F_\infty)$.

$$\text{Let } \epsilon' = (4+B)\epsilon$$

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \left[\int h(\theta-D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta-D) \ell(\underline{x}|\theta) dF_n(\theta) \right] \\ &= \lim_{\epsilon' \downarrow 0} \lim_{n \rightarrow \infty} \left[\int h(\theta-D_\infty(\epsilon')) \ell(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta-D) \ell(\underline{x}|\theta) dF_n(\theta) \right] \\ &= 0. \quad (\text{by strongly stable of definition III}). \end{aligned}$$

We thus show the triple is also strongly stable by definition IV'. ■

We have gone through all possible cases about the triple $(h, \ell(\underline{x}|\theta), F_\infty)$, where h is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. We show if a triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is unstable by definition III' then the triple is also unstable by definition IV', and if a triple is weakly stable by definition III' then the triple is weakly stable by definition IV'. Finally, we go through every case to show if a triple is strongly stable by definition III' then the triple is strongly stable by definition IV'. Thus we have the following theorems.

Theorem 3.3: For the estimation or prediction problem, a triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is strongly (weakly) stable by definition III' iff it is strongly (weakly) stable by definition IV'.

And by the discussion at the beginning of this chapter we have:

Theorem 3.4: For the estimation or prediction problem, definitions I, II, III, IV, I', II', III' and IV' are all equivalent to each other.

Chapter 4. Necessary and Sufficient Conditions for Stability in the Estimation Case

4.1 When Loss Function is Uniformly Bounded or Uniformly Bounded on One Side and Unbounded on the Other Side.

In this chapter, we study conditions for a triple to be stable. We assume $L_{\infty}(\theta, D) = h(\theta - D)$, where $h(x)$ is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. Under these conditions, we showed in Chapter 3 that definitions I, II, III, IV, I', II', III', and IV' are all equivalent; i.e., a triple is strongly (weakly) stable by definition I iff the triple is strongly (weakly) stable by any other definitions. Among these definitions, definition III' is the easiest, so in this chapter we thus use definition III' as our criterion for stability. As in Chapter 3, in this chapter we consider first the case that $h(x)$ is uniformly bounded, then consider $h(x)$ uniformly bounded on one side and unbounded on the other side, and finally we discuss those $h(x)$ that are unbounded on both sides.

By Theorem 3.2, we see that if $h(x)$ is uniformly bounded, then for any $\mathcal{L}(\underline{x}|\underline{q})$ and F_{∞} , the triple $(h, \mathcal{L}(\underline{x}|\underline{q}), F_{\infty})$ is strongly stable by any definition.

Next, we consider those $h(x)$ uniformly bounded on one side and unbounded on the other side. Without loss of generality, we assume $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $\lim_{x \rightarrow -\infty} h(x) = A < +\infty$. The following lemma gives a sufficient condition for a triple to be stable.

Lemma 4.1. Suppose $\lim_{x \rightarrow -\infty} h(x) = +\infty$ and $\lim_{x \rightarrow -\infty} h(x) = A < +\infty$.

Also suppose that for all $\epsilon > 0$ there exists $D_\infty(\epsilon)$ such that $h(\theta - D_\infty(\epsilon))l(\underline{x}|\theta)$ is uniformly bounded. Then for these $D_\infty(\epsilon)$ there exists C_0 such that for all $F_n \xrightarrow{W} F_\infty$

$$(4.1) \quad \lim_{n \rightarrow \infty} \sup \left[\int h(\theta - D_\infty(\epsilon))l(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - D_n(\epsilon))l(\underline{x}|\theta) dF_n(\theta) \right] < C_0 \epsilon,$$

where $D_n(\epsilon)$ is an ϵ -optimal decision for the triple $(h, l(\underline{x}|\theta), F_n)$.

Proof: Let $R = A(\int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_\infty(\theta))$, $C_1 = \int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_\infty(\theta)$.

By the assumption $\lim_{x \rightarrow +\infty} h(x) = +\infty$, we can see that $D_n(\epsilon)$ is uniformly bounded below. And we can find a and N , such that $n > N, D_n(\epsilon) > a$.

Thus we assume that when n is large, $D_n(\epsilon) \in [a, \infty)$. Without loss of generality we assume $D_\infty(\epsilon) \in [a, \infty)$.

We can find $\epsilon < 0$, such that $h(\epsilon) > A - \epsilon$ and we can find b such that 1) $\int_b^\infty h(\theta - a)l(\underline{x}|\theta) dF_\infty(\theta) < \epsilon$; 2) b is a continuous point of $F_\infty(\theta)$ and $F_\infty(b + \epsilon) > 1 - \epsilon$; 3) $b > D_\infty(\epsilon)$.

And let $c < a$, c is a continuous point of $F_\infty(\theta)$ and

$$(4.2) \quad \int_{-\infty}^c h(\theta - b)l(\underline{x}|\theta) dF_\infty(\theta) < \epsilon.$$

Now we will prove (4.1) is true when $D_n(\epsilon) \in [a, b]$ and later prove (4.1) is true for $D_n(\epsilon) \in (b, \infty)$.

By 1) and (4.2) we know if $D_n(\epsilon) \in [a, b]$,

$$(4.3) \quad \int_{-\infty}^c + \int_b^\infty h(\theta - D_n(\epsilon))l(\underline{x}|\theta) dF_\infty(\theta) < \int_{-\infty}^c h(\theta - b)l(\underline{x}|\theta) dF_\infty(\theta) + \int_b^\infty h(\theta - a)l(\underline{x}|\theta) dF_\infty(\theta) < 2\epsilon.$$

By Lemma 3.3, we can find N_0 such that for all $n > N_0$ and

for all $u, v \in [a, b]$, we have

$$\begin{aligned}
 (4.4) \quad & \int_c^b h(\theta - u) \ell(\underline{x} | \theta) dF_n(\theta) - \int_c^b h(\theta - v) \ell(\underline{x} | \theta) dF_n(\theta) \\
 & \leq \int_c^b h(\theta - u) \ell(\underline{x} | \theta) dF_\infty(\theta) - \int_c^b h(\theta - v) \ell(\underline{x} | \theta) dF_\infty(\theta) + \epsilon.
 \end{aligned}$$

Because $h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta)$ is uniformly bounded, there exists N_1 such that for all $n > N_1$,

$$\begin{aligned}
 (4.5) \quad & \int_{-\infty}^c h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_n(\theta) + \int_b^\infty h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_n(\theta) \\
 & < \int_{-\infty}^c h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_\infty(\theta) + \int_b^\infty h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_\infty(\theta) + \epsilon.
 \end{aligned}$$

Now we are ready to prove (4.1) is true for all $D \in [a, b]$. When $n > \max(N_0, N_1)$,

$$\begin{aligned}
 & \int_{-\infty}^\infty (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) \ell(\underline{x} | \theta) dF_n(\theta) \\
 & \leq \int_{-\infty}^\infty h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_n(\theta) - \int_c^b h(\theta - D_n(\epsilon)) \ell(\underline{x} | \theta) dF_n(\theta) \\
 & \leq \int_{-\infty}^c + \int_b^\infty h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_n(\theta) + \int_c^b (h(\theta - D_\infty(\epsilon)) \\
 & \quad - h(\theta - D_n(\epsilon))) \ell(\underline{x} | \theta) dF_\infty(\theta) + \epsilon
 \end{aligned}$$

(by (4.4))

$$\begin{aligned}
 & \leq \int_{-\infty}^\infty h(\theta - D_\infty(\epsilon)) \ell(\underline{x} | \theta) dF_\infty(\theta) - \int_c^b h(\theta - D_n(\epsilon)) \ell(\underline{x} | \theta) dF_\infty(\theta) + 2\epsilon. \\
 & \hspace{25em} \text{(by (4.5))}
 \end{aligned}$$

$$\leq \int_{-\infty}^{\infty} h(\theta - D_{\infty}(\epsilon)) \ell(x|\theta) dF_{\infty}(\theta) - \int_{-\infty}^{\infty} h(\theta - D_n(\epsilon)) \ell(x|\theta) dF_{\infty}(\theta) + 4\epsilon. \\ \text{(by (4.3))}$$

$$\leq \epsilon + 4\epsilon = 5\epsilon. \quad \text{(by definition of } D_{\infty}(\epsilon))$$

Now we have to consider $D_n(\epsilon) \in (b, \infty)$.

By uniform boundedness of $h(\theta - D_{\infty}(\epsilon)) \ell(x|\theta)$ there exists N_2 , such that for all $n > N_2$,

$$(4.6) \quad \int h(\theta - D_{\infty}(\epsilon)) \ell(x|\theta) dF_n(\theta) < \int h(\theta - D_{\infty}(\epsilon)) \ell(x|\theta) dF_{\infty}(\theta) + \epsilon.$$

$$\lim_{D \rightarrow \infty} \int h(\theta - D) \ell(x|\theta) dF_{\infty}(\theta) = \left(\int \ell(x|\theta) dF_{\infty}(\theta) \right) \left(\lim_{D \rightarrow \infty} \int h(\theta - D) dP_{\infty}(\theta) \right) = R.$$

By definition of $D_{\infty}(\epsilon)$ and the above equality, we know the right hand side of (4.6) is less than $R + 2\epsilon$. Since $P_n(\theta) \xrightarrow{w} P_{\infty}(\theta)$, we can find N_3 such that for all $n > N_3$, $P_n(b + \epsilon) > 1 - 2\epsilon$. And we can find N_4 such that for all $n > N_4$, $\int \ell(x|\theta) dF_n(\theta) \geq C_1 - \epsilon$.

Then for all $n > \max(N_2, N_3, N_4)$ and $D_n(\epsilon) \in (b, \infty)$,

$$\begin{aligned} & \int_{-\infty}^{\infty} h(\theta - D_n(\epsilon)) \ell(x|\theta) dF_n(\theta) \\ &= \left(\int \ell(x|\theta) dF_n(\theta) \right) \left(\int_{-\infty}^{\infty} h(\theta - D_n(\epsilon)) dP_n(\theta) \right) \\ &\geq (C_1 - \epsilon) \int_{-\infty}^{b+\epsilon} h(\theta - D_n(\epsilon)) dP_n(\theta) \\ &\geq (C_1 - \epsilon) h(b + \epsilon - b) \cdot \int_{-\infty}^{b+\epsilon} dP_n(\theta) \\ &\geq (C_1 - \epsilon) (A - \epsilon) (1 - 2\epsilon) \\ &\geq A \cdot C_1 - (A + C_1 + 2AC_1) \epsilon = R - (A + C_1 + 2AC_1) \epsilon. \quad \text{Then} \end{aligned}$$

$$\int h(\theta - D_{\infty}(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) \leq \\ R + 2\epsilon - R + (A + C_1 + 2AC_1)\epsilon = (2 + A + C_1 + 2AC_1)\epsilon$$

Let $C_0 = \max(5, A + C_1 + 2AC_1 + 2)$. Then

$$\lim_{n \rightarrow \infty} [\int h(\theta - D_{\infty}(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - D_n(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta)] < C_0 \epsilon. \quad \blacksquare$$

The following theorem summarizes all results related to loss functions that are uniformly bounded on one side and unbounded on the other side.

Theorem 4.1: Suppose $\lim_{x \rightarrow +\infty} h(x) = +\infty$, $\lim_{x \rightarrow -\infty} h(x) = A < +\infty$, then

(a) if there exists $\epsilon_0 > 0$, such that for all $D_{\infty}(\epsilon)$, $h(\theta - D_{\infty}(\epsilon)) \ell(\underline{x}|\theta)$ as a function of θ , is uniformly bounded, then $(h, \ell(\underline{x}|\theta), F_{\infty})$ is strongly stable.

(b) if there exists $\epsilon_0 > 0$ such that for all $D(\epsilon_0)$, $h(\theta - D_{\infty}(\epsilon_0)) \ell(\underline{x}|\theta)$ is unbounded, then $(h, \ell(\underline{x}|\theta), F_{\infty})$ is unstable.

(c) for all $\epsilon > 0$, there exists $D_{\infty}^1(\epsilon)$ and $D_{\infty}^2(\epsilon)$ satisfying (2.9), such that $h(\theta - D_{\infty}^1(\epsilon)) \ell(\underline{x}|\theta)$ is uniformly bounded and $h(\theta - D_{\infty}^2(\epsilon)) \ell(\underline{x}|\theta)$ is unbounded, then $(h, \ell(\underline{x}|\theta), F_{\infty})$ is weakly stable and $D_{\infty}^1(\epsilon)$ is the stabilizing decision.

Proof: (a), (c) follow directly from Lemmas 4.1 and 3.3. For (b), let $D_S = \sup\{D_{\infty}(\epsilon_0)\}$, where $\{D_{\infty}(\epsilon_0)\}$ is the set of decisions satisfying (2.9). Then either (i) D_S is a finite number or (ii) $D_S = +\infty$. In the following we find θ_n for each case.

(i) We know $h(\theta - D_S) \ell(\underline{x}|\theta)$ is unbounded, thus we can find $\theta_n > D_S$ such that $h(\theta_n - D_S) \ell(\underline{x}|\theta_n) > n^2$. Then for all $D_{\infty}(\epsilon_0)$, $h(\theta_n - D_{\infty}(\epsilon_0)) \ell(\underline{x}|\theta_n) > n^2$.

(ii) In this case, for any D we have $h(\theta-D)\ell(x|\theta)$ is unbounded. Thus let θ_n satisfy $h(\theta_n-n)\ell(x|\theta_n) > n^2$. Then for all $D_\infty(\epsilon_0)$, as $n > D_\infty(\epsilon_0)$, $h(\theta_n-D_\infty(\epsilon_0))\ell(x|\theta_n) > n^2$.

Now let $F_n(\theta) = (1 - \frac{1}{n})F_\infty(\theta) + \frac{1}{n}J_n(\theta)$, where $J_n(\theta)$ is defined as

$$J_n(\theta) = \begin{cases} 0 & \theta < \theta_n \\ 1 & \theta \geq \theta_n. \end{cases}$$

Then for all $D_\infty(\epsilon_0)$,

$$\begin{aligned} & \int h(\theta-D_\infty(\epsilon_0))\ell(x|\theta)dF_n(\theta) - \inf_D \int h(\theta-D)\ell(x|\theta)dF_n(\theta) \\ & \geq (1 - \frac{1}{n})(\int h(\theta-D_\infty(\epsilon_0))\ell(x|\theta)dF_\infty(\theta) - \int h(\theta-\theta_n)\ell(x|\theta)dF_\infty(\theta)) \\ & \quad + \frac{1}{n} h(\theta_n-D_\infty(\epsilon_0))\ell(x|\theta_n) \\ & \geq \frac{1}{n} \cdot n^2 - \int h(\theta-\theta_n)\ell(x|\theta)dF_\infty(\theta). \end{aligned}$$

In the proof of Lemma 3.3, we know $\int h(\theta-\theta_n)\ell(x|\theta)dF_\infty(\theta)$ is bounded by a constant which is independent of n . Thus for all $D_\infty(\epsilon_0)$,

$$\lim_{n \rightarrow \infty} [\int h(\theta-D_\infty(\epsilon_0))\ell(x|\theta)dF_n(\theta) - \inf_D \int h(\theta-D)\ell(x|\theta)dF_n(\theta)] = +\infty.$$

And for all $\epsilon < \epsilon_0$, we know $D_\infty(\epsilon) \in \{D_\infty(\epsilon_0)\}$. So

$$\lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} (\int h(\theta-D_\infty(\epsilon))\ell(x|\theta)dF_n(\theta) - \inf_D \int h(\theta-D)\ell(x|\theta)dF_n(\theta)) = \infty.$$

And the triple is unstable by definition III'. ■

4.2 The Case $\lim_{x \rightarrow +\infty} h(x) = +\infty$.

In this section, we study those triples $(h, \ell(x|\theta), F_\infty)$ whose loss functions satisfy $\lim_{x \rightarrow +\infty} h(x) = +\infty$. Under this assumption, $D_\infty(\epsilon)$ must be uniformly bounded. Define $\{D_\infty(\epsilon)\}$ to be the set of all decisions satisfying (2.9). Let $D_m(\epsilon) = \inf\{D_\infty(\epsilon)\}$ and $D_s(\epsilon) = \sup\{D_\infty(\epsilon)\}$. These definitions are used throughout this section. The following lemma gives sufficient condition for a triple to be unstable.

Lemma 4.2. If there exists $\epsilon > 0$ such that $\forall D_\infty(\epsilon)$, we can find a D_0 (D_0 depends on $D_\infty(\epsilon)$) such that $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_0))$, as a function of θ , is not uniformly bounded above (u.b.a.), then $(h, \ell(x|\theta), F_\infty)$ is unstable.

Proof: We discuss the problem in three parts. In the first part we can find some fixed decision D_f (independent of $D_\infty(\epsilon)$) such that for all $D_\infty(\epsilon)$, $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_f))\ell(x|\theta)$ is not uniformly bounded above (u.b.a.). In the second part, we can find two fixed decisions D_f and D_g such that for all $D_\infty(\epsilon)$ either $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_f))\ell(x|\theta)$ or $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_g))\ell(x|\theta)$ is not u.b.a. In the third part we discuss the remaining cases.

(a) If there exists D_0 such that (i) $D_0 \geq D_s(\epsilon)$ and $(h(\theta - D_s(\epsilon)) - h(\theta - D_0))\ell(x|\theta)$ is not u.b.a. or (ii) $D_0 \leq D_m(\epsilon)$ and $(h(\theta - D_m(\epsilon)) - h(\theta - D_0))\ell(x|\theta)$ is not u.b.a., then the triple is unstable.

For (i) let θ_1 satisfy $\theta_1 > D_0$ and $(h(\theta_1 - D_s(\epsilon)) - h(\theta_1 - D_0))\ell(x|\theta_1) > i^2$, and for (ii) let θ_1 satisfy $\theta_1 < D_0$ and $(h(\theta_1 - D_m(\epsilon)) - h(\theta_1 - D_0))\ell(x|\theta_1) > i^2$.

Then for both (i) and (ii) we have:

$$(h(\theta_1 - D_\infty(\epsilon)) - h(\theta_1 - D_0))\ell(x|\theta_1) > 1^2.$$

Let $F_n(\theta) = (1 - \frac{1}{n})F_\infty(\theta) + \frac{1}{n}J_n(\theta)$, where $J_n(\theta)$ is defined as

$$J_n(\theta) = \begin{cases} 0 & \theta < \theta_n \\ 1 & \theta \geq \theta_n. \end{cases}$$

Now for any $D_\infty(\epsilon)$,

$$\begin{aligned} & \int h(\theta - D_\infty(\epsilon))\ell(x|\theta)dF_n(\theta) - \inf_D \int h(\theta - D)\ell(x|\theta)dF_n(\theta) \\ & \geq \int h(\theta - D_\infty(\epsilon))\ell(x|\theta)dF_n(\theta) - \int h(\theta - D_0)\ell(x|\theta)dF_n(\theta) \\ & \geq \frac{n-1}{n}(\int (h(\theta - D_\infty(\epsilon)) - h(\theta - D_0))\ell(x|\theta)dF_\infty(\theta)) + \frac{1}{n} \cdot n^2. \end{aligned}$$

When $n \rightarrow \infty$, the right hand side goes to infinity. Thus

$$\lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \sup [\int h(\theta - D_\infty(\epsilon))\ell(x|\theta)dF_n(\theta) - \inf_D \int h(\theta - D)\ell(x|\theta)dF_n(\theta)] = +\infty,$$

and the triple is unstable.

(b) If we can find d , D_0 , and D_1 such that $D_0 > d$, $D_1 < d$ and both $(h(\theta - d) - h(\theta - D_0))\ell(x|\theta)$ and $(h(\theta - d) - h(\theta - D_1))\ell(x|\theta)$ are not u.b.a., then the triple is unstable. We consider this in two cases.

(i) if $d \geq D_S(\epsilon)$ or $d \leq D_m(\epsilon)$, then by (a) the triple is unstable.

(ii) if $D_m(\epsilon) < d < D_S(\epsilon)$.

Because $D_0 > d$, so we can find $\theta_{01} > D_0$ such that

$$(h(\theta_{01} - d) - h(\theta_{01} - D_0))\ell(x|\theta_{01}) > 1^2.$$

Similarly, we can also find $\theta_{11} < D_1$ such that

$$(h(\theta_{11} - d) - h(\theta_{11} - D_1))\ell(x|\theta_{11}) > 1^2.$$

Let $G_n(\theta) = (1 - \frac{1}{n}) F_\infty(\theta) + \frac{1}{n} J_n(\theta)$

$$\text{where } J_n(\theta) = \begin{cases} 0 & \theta < \theta_{on} \\ 1 & \theta \geq \theta_{on} \end{cases}.$$

And let $H_n(\theta) = (1 - \frac{1}{n}) F_\infty(\theta) + \frac{1}{n} K_n(\theta)$

$$\text{where } K_n(\theta) = \begin{cases} 0 & \theta < \theta_{ln} \\ 1 & \theta \geq \theta_{ln} \end{cases}.$$

Now for all $D \in [d, D_s(\epsilon)]$ we have

$$(h(\theta_{11}-D) - h(\theta_{11}-D_1))\ell(x|\theta_{11}) > 1^2.$$

And all $D \in [D_m(\epsilon), d)$ we have

$$(h(\theta_{01}-D) - h(\theta_{01}-D_0))\ell(x|\theta_{01}) > 1^2.$$

The sequence $G_1, H_1, G_2, H_2, G_3, H_3, \dots$ converges in distribution to $F_\infty(\theta)$. And by the same arguments as in (1)

$$\lim_{n \rightarrow \infty} \sup \left[\int h(\theta - D_\infty(\epsilon)) \ell(x|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \ell(x|\theta) dF_n(\theta) \right] = +\infty.$$

Here we should mention, that when n is odd let $D = D_0$ and when n is even let $D = D_1$ to achieve the result.

Thus the triple is unstable.

(c) From (a) and (b) we can see the case that is left is that for all $D_m(\epsilon) \leq D_\infty(\epsilon) \leq D_s(\epsilon)$, we can find $D_0 < D_\infty(\epsilon)$ or $D_1 > D_\infty(\epsilon)$ (but not both), where D_0 and D_1 depends on $D_\infty(\epsilon)$, such that $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_0))\ell(x|\theta)$ or $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_1))\ell(x|\theta)$ is unbounded. Also for all $D > D_s(\epsilon)$, $(h(\theta - D_s(\epsilon)) - h(\theta - D))\ell(x|\theta)$ is u.b.a. and there exists $D_0 < D_s(\epsilon)$ such that $(h(\theta - D_s(\epsilon)) - h(\theta - D_0))\ell(x|\theta)$ is not u.b.a. Also for all $D < D_m(\epsilon)$, $(h(\theta - D_s(\epsilon)) - h(\theta - D))\ell(x|\theta)$ is u.b.a. and there exists $D_0 > D_m(\epsilon)$

such that $(h(\theta - D_m(\epsilon)) - h(\theta - D_o)) \mathbb{I}(\underline{x}|\theta)$ is not u.b.a..

Without loss of generality, we assume for all $D \in [D_m(\epsilon), D_s(\epsilon)]$, then $D \in \{D_\infty(\epsilon)\}$.

Now we are going to argue there exists a unique decision $D_t [D_m(\epsilon), D_s(\epsilon)]$ such that (1) for all $D \in (D_t, D_s(\epsilon)]$ we can find $d_o < D$ (d_o depends on D), such that $(h(\theta - D) - h(\theta - d_o)) \mathbb{I}(\underline{x}|\theta)$ is not u.b.a., but for all $d > D$ $(h(\theta - D) - h(\theta - d)) \mathbb{I}(\underline{x}|\theta)$ is u.b.a., and (2) for all $D \in (D_m(\epsilon), D_t)$ we can find $d_1 > D$ (d_1 depends on D) such that $(h(\theta - D) - h(\theta - d_1)) \mathbb{I}(\underline{x}|\theta)$ is not u.b.a., but for all $d < D$ $(h(\theta - D) - h(\theta - d)) \mathbb{I}(\underline{x}|\theta)$ is u.b.a.

If there are more than one point satisfy the above conditions, say D_t and d_t and let $D_t > d_t$. Then any point D , $D_t > D > d_t$, we can find d_1 and d_o such that $(h(\theta - D) - h(\theta - d_1)) \mathbb{I}(\underline{x}|\theta)$ and $(h(\theta - D) - h(\theta - d_o)) \mathbb{I}(\underline{x}|\theta)$ are not u.b.a.. This contradicts our assumption.

Now we discuss this unique point D_t . $D_t \in \{D_\infty(\epsilon)\}$, then we can find D_1 such that $(h(\theta - D_t) - h(\theta - D_1)) \mathbb{I}(\underline{x}|\theta)$ is not u.b.a. And without loss of generality we assume $D_1 > D_t$.

Let θ_{o1} satisfy $\theta_{o1} > D_1$ and $(h(\theta_{o1} - D_t) - h(\theta_{o1} - D_1)) \mathbb{I}(\underline{x}|\theta_{o1}) > i^2$. Then for all $D \in [D_m(\epsilon), D_t]$, $(h(\theta_{o1} - D) - h(\theta_{o1} - D_1)) \mathbb{I}(\underline{x}|\theta_{o1}) > i^2$. And let $G_1(\theta) = (1 - \frac{1}{i}) F_\infty(\theta) + \frac{1}{i} J_1(\theta)$, where $J_1(\theta)$ is defined as

$$J_1(\theta) = \begin{cases} 0 & \theta < \theta_{01} \\ 1 & \theta \geq \theta_{01} \end{cases}.$$

And let $D_k = D_t + \frac{1}{k}$, we can find θ_{1k} and d_k such that $\theta_{1k} < d_k < D_k$ and $(h(\theta_{1k}-D_k) - h(\theta_{1k}-d_k))\mathcal{L}(\underline{x}|\theta_{1k}) > k^2$. And for all $D > D_k$ $(h(\theta_{1k}-D) - h(\theta_{1k}-d_k))\mathcal{L}(\underline{x}|\theta_{1k}) > k^2$.

Let $H_k(\theta) = (1 - \frac{1}{k})F_\infty + \frac{1}{k}K_k(\theta)$, where $K_k(\theta)$ is defined as

$$K_k(\theta) = \begin{cases} 0 & \theta < \theta_{1k} \\ 1 & \theta \geq \theta_{1k} \end{cases}.$$

Then $G_1, H_1, G_2, H_2, \dots \xrightarrow{w} F_\infty$, and

$$\lim_{n \rightarrow +\infty} [\int h(\theta - D_\infty(\epsilon))\mathcal{L}(\underline{x}|\theta) dF_n(\theta) - \inf_b \int h(\theta - D)\mathcal{L}(\underline{x}|\theta) dF_n(\theta)] = +\infty,$$

where $F_{2n-1} = G_n$ and $F_{2n} = H_n$.

There in all of these cases, $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable. ■

The following lemma studies a sufficient condition for a triple to be strongly stable. We should mention here that Lemma 3.7 gives us a sufficient condition for a triple to be unstable.

Lemma 4.3 Suppose there exists $\epsilon_0 > 0$ such that for all $D_\infty(\epsilon_0)$ and D , $(h(\theta - D_\infty(\epsilon_0)) - h(\theta - D))\mathcal{L}(\underline{x}|\theta)$ is uniformly bounded above, and suppose there exists two constants a and b , and N such that for all $n > N$, $D_n(\epsilon) \in [a, b]$. Then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable.

Proof: Our object is to show if for all $F_n(\theta) \xrightarrow{W} F_\infty(\theta)$, we can find N such that for all $n > N$ and $D_\infty(\epsilon) \in [a, b]$ we have:

$$(4.7) \quad \int h(\theta - D_\infty(\epsilon)) l(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) l(\underline{x}|\theta) dF_n(\theta) < B(\epsilon),$$

$$\text{and } \lim_{\epsilon \downarrow 0} B(\epsilon) = 0$$

In order to free us from considering about end points, let $\epsilon_1 = \epsilon_0/2$, then $D_m(\epsilon_0) < D_m(\epsilon_1) < D_s(\epsilon_1) < D_s(\epsilon_0)$.

Let c and d be continuous points of F_∞ , such that $c < a < b < d$,

$$(4.8) \quad \int_d^\infty h(\theta - a) l(\underline{x}|\theta) dF_\infty < \epsilon \quad \text{and}$$

$$(4.9) \quad \int_{-\infty}^c h(\theta - b) l(\underline{x}|\theta) dF_\infty(\theta) < \epsilon$$

Our object is to show (4.7). We first consider $\theta \in (-\infty, c) \cup (d, \infty)$. By (4.8) and (4.9), we know $\forall D \in [a, b]$

$$\int_d^\infty h(\theta - D) l(\underline{x}|\theta) dF_\infty(\theta) < \epsilon \quad \text{and}$$

$$\int_{-\infty}^c h(\theta - D) l(\underline{x}|\theta) dF_\infty(\theta) < \epsilon$$

Since $(h(\theta - D_m(\epsilon_1)) - h(\theta - b)) l(\underline{x}|\theta)$ is uniformly bounded above then by the Convergence Theorem (Loeve, 1963), there exists N that for all $n > N$,

$$\int_d^\infty (h(\theta - D_m(\epsilon_1)) - h(\theta - b)) l(\underline{x}|\theta) dF_n < \int_d^\infty (h(\theta - r$$

and

$$\int_{-\infty}^c (h(\theta - D_s(\epsilon_1)) - h(\theta - a)) \mathbb{L}(\underline{x}|\theta) dF_n < \int_{-\infty}^c (h(\theta - D_s(\epsilon_1)) - h(\theta - a)) \mathbb{L}(\underline{x}|\theta) dF_{\infty} + \epsilon.$$

By these two inequalities, we can see that $\forall n > N_0$, and for all $t \in [D_i(\epsilon_1), D_s(\epsilon_1)]$, and for all $D \in [a, b]$,

$$(4.10) \quad \int_d^{+\infty} ((h(\theta - t) - h(\theta - D)) \mathbb{L}(\underline{x}|\theta) dF_n < \int_d^{+\infty} (h(\theta - D_m(\epsilon_1)) - h(\theta - b)) \mathbb{L}(\underline{x}|\theta) dF_n \\ < \int_d^{\infty} (h(\theta - D_m(\epsilon_1)) - h(\theta - b)) dF_{\infty} + \epsilon < 2\epsilon,$$

and

$$(4.11) \quad \int_{-\infty}^c (h(\theta - t) - h(\theta - D)) \mathbb{L}(\underline{x}|\theta) dF_n < \int_{-\infty}^c (h(\theta - D_s(\epsilon_1)) - h(\theta - a)) \mathbb{L}(\underline{x}|\theta) dF_{\infty} + \epsilon \\ < 2\epsilon.$$

Now we consider $\theta \in [c, d]$.

By Lemma 3.3 we can find N_1 such that $n > N_1$ and for all

d_1 and $d_2 \in [a, b]$,

$$(4.12) \quad \int_c^d (h(\theta - d_1) - h(\theta - d_2)) \mathbb{L}(\underline{x}|\theta) dF_n(\theta) \leq \int_c^d (h(\theta - d_1) - h(\theta - d_2)) \mathbb{L}(\underline{x}|\theta) dF_{\infty}(\theta) + \epsilon$$

Let $N = \max \{N_0, N_1\}$, then for all $n > N$ and for all $D_{\infty}(\epsilon)$,

$D \in [a, b]$

$$\int h(\theta - D_{\infty}(\epsilon)) \mathbb{L}(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - D) \mathbb{L}(\underline{x}|\theta) dF_n(\theta) \\ \leq \int_c^d h(\theta - D_{\infty}(\epsilon)) \mathbb{L}(\underline{x}|\theta) dF_{\infty}(\theta) - \int_c^d h(\theta - D) \mathbb{L}(\underline{x}|\theta) dF_{\infty} + \epsilon \\ + \int_{-\infty}^c h(\theta - D_s(\epsilon_1)) \mathbb{L}(\underline{x}|\theta) dF_n(\theta) - \int_{-\infty}^c h(\theta - a) \mathbb{L}(\underline{x}|\theta) dF_n(\theta)$$

$$\begin{aligned}
& + \int_d^\infty h(\theta - D_m(\epsilon_1)) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) - \int_{-\infty}^c h(\theta - b) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) \quad (\text{by (4.12)}) \\
& \leq \int_c^d h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - \int_c^d h(\theta - D) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + 5\epsilon \quad (\text{by (4.10) and (4.11)}) \\
& \leq \int_{-\infty}^\infty h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) - \int_{-\infty}^\infty h(\theta - D) \mathcal{L}(\underline{x}|\theta) dF_\infty(\theta) + 6\epsilon \\
& \leq 7\epsilon.
\end{aligned}$$

Thus we have shown

$$\limsup_{n \rightarrow \infty} \left[\int h(\theta - D_\infty(\epsilon)) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) - \inf_D \int h(\theta - D) \mathcal{L}(\underline{x}|\theta) dF_n(\theta) \right] < 7\epsilon.$$

And this implies that the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable. ■

From Lemmas 4.2 and 4.3 we can easily see the following theorem.

Theorem 4.2 Suppose there exists $[a, b]$ and N such that for all $n > N$, $D_n(\epsilon) \in [a, b]$, then

(a) if there exists $\epsilon_0 > 0$ such that for all $D_\infty(\epsilon_0)$ and D , $(h(\theta - D_\infty(\epsilon)) - h(\theta - D)) \mathcal{L}(\underline{x}|\theta)$ is uniformly bounded above, then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable.

(b) If there exists $\epsilon_0 > 0$ such that for all $D_\infty(\epsilon_0)$, we can find D (depends on $D_\infty(\epsilon)$) such that $(h(\theta - D_\infty(\epsilon)) - h(\theta - D)) \mathcal{L}(\underline{x}|\theta)$ is not bounded above, then the triple is unstable.

(c) For all $\epsilon > 0$, there exists $D_\infty^1(\epsilon)$ and $D_\infty^2(\epsilon) \in \{D_\infty(\epsilon)\}$ such that $\forall D$, $(h(\theta - D_\infty^1(\epsilon)) - h(\theta - D)) \mathcal{L}(\underline{x}|\theta)$ is uniformly bounded above and there exists D_0 (depends on $D_\infty^2(\epsilon)$) such that

$(h(\theta - D_\infty^2(\epsilon)) - h(\theta - D_0))\lambda(\underline{x}|\theta)$ is not uniformly bounded above, then the triple is weakly stable and $D_\infty^1(\epsilon)$ is the stabilizing decision.

Proof: Direct use of Lemmas 4.2 and 4.3. ■

The importance of this theorem is that this theorem makes Theorem 4.3 easier to understand and formulate.

From the above lemmas and theorem, it seems that whether the triple $(h, \lambda(\underline{x}|\theta), F_\infty)$ is strongly stable or not depends on h and $\lambda(\underline{x}|\theta)$ only. This is not true because $D_\infty(\epsilon)$ depends on $F_\infty(\theta)$. The following example shows this explicitly. The following example is interesting and important. In Chapter 5, we study the estimation problems under the additional assumption $\lambda(\underline{x}|\theta) \equiv 1$. Under these conditions, we can prove (i) a pair (h, F_∞) cannot be weakly stable; (ii) For any $F_\infty(\theta)$ and $G_\infty(\theta)$, (h, F_∞) is strongly stable iff (h, G_∞) is strongly stable. However, the following example shows these two results are not necessarily true when $\lambda(\underline{x}|\theta) \neq 1$.

Example 4.1. Suppose $\lambda(\underline{x}|\theta)$ and $h(t)$ are defined by:

$$h(t) = \begin{cases} |t| & t < 1 \\ t^t & t \geq 1 \end{cases}$$

$$\lambda(\underline{x}|\theta) = \begin{cases} e^{\theta+1} & \theta < -1 \\ 1 & -1 \leq \theta \leq 1 \\ \frac{1}{h(\theta)} & \theta > 1 \end{cases} .$$

(a) If there exists $\epsilon_0 > 0$ such that all $D_\infty(\epsilon_0) \geq 0$, then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable. (One example is $F_\infty(\theta)$ is uniformly distributed in $(0,1)$.)

(b) If there exists $\epsilon_0 > 0$ such that all $D_\infty(\epsilon_0) < 0$, then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable. (One example is $F_\infty(\theta)$ is uniformly distributed in $(-1,0)$.)

(c) If for all $\epsilon > 0$, there exists $D_\infty^1(\epsilon) \geq 0$ and $D_\infty^2(\epsilon) < 0$, then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is weakly stable and $D_\infty^1(\epsilon)$ is the stabilizing decision. (One example is $F_\infty(\theta)$ is uniformly distributed in $(-\frac{1}{2}, \frac{1}{2})$.)

Proof: First we show that for all $D \geq 0$, $h(\theta-D)\mathcal{L}(\underline{x}|\theta)$ is uniformly bounded. When $\theta > D$, $h(\theta-D)\mathcal{L}(\underline{x}|\theta) \leq h(\theta)\mathcal{L}(\underline{x}|\theta) \leq 1$

$\theta < -1$ $h(\theta-D)\mathcal{L}(\underline{x}|\theta) = |-\theta+D|e^{\theta+1}$, this is also uniformly bounded.

For $D < 0$, we show $h(\theta-D)\mathcal{L}(\underline{x}|\theta)$ is unbounded. Let $c = |D|$, $\theta > 0$, $h(\theta-D)\mathcal{L}(\underline{x}|\theta) = h(\theta+c)\mathcal{L}(\underline{x}|\theta) = (\theta+c)^{\theta+c}/\theta^\theta \geq (\theta+c)^c \rightarrow +\infty$ as $\theta \rightarrow +\infty$.

And since $h(\theta-0)\mathcal{L}(\underline{x}|\theta)$ is uniformly bounded, $\lim_{n \rightarrow \infty} \int h(\theta)\mathcal{L}(\underline{x}|\theta) dF_n(\theta) = \int h(\theta)\mathcal{L}(\underline{x}|\theta) dF_\infty(\theta)$. Thus $D_n(\epsilon)$ is uniformly bounded.

So this example satisfies the conditions in Theorem 4.2 and the results follow directly from Theorem 4.2. ■

In Lemma 3.7, we show if one can find $D_n(\epsilon) \rightarrow \infty$, then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable. In Theorem 4.2, we discussed all cases under the assumption that $D_n(\epsilon)$ cannot go to infinity.

The next theorem shows necessary and sufficient conditions that $D_n(\epsilon)$ cannot go to infinity.

Theorem 4.3. (a) Suppose there exists r, d, t_0 and D_0 , such that t_0 is a continuous point of $P_\infty(\theta)$ and $P_\infty(t_0) > 0$ and for all $D > D_0$

$$(4.13) \quad \frac{h(t_0 - D)}{\sup_{\theta > E} (h(\theta - d) - h(\theta - D))I(\underline{x}|\theta)} > r$$

where E satisfies $d < E < D$ and $h(E - d) = h(E - D)$. Then $D_n(\epsilon)$ cannot go to $+\infty$.

(b) Suppose for all r, d, t, D_0 , where t is a continuous point of $P_\infty(\theta)$, $P_\infty(t) > 0$, there exists $D > D_0$ such that

$$(4.14) \quad \frac{h(t - D)}{\sup_{\theta > E} (h(\theta - d) - h(\theta - D))I(\underline{x}|\theta)} < r$$

where E is defined in (a).

Then the triple is unstable.

Proof: (a) Let t_1 be a continuous point of $P_\infty(\theta)$ and $P_\infty(t_1) < P_\infty(t_0)$. In this part, we show d is the decision that prevents $D_n(\epsilon)$ from going to infinity. Our object is to show there exists $b > 0, c_0 > 0$ and N such that for all $n > N, D > b$ we have

$$\int h(\theta - d)I(\underline{x}|\theta)dF_n(\theta) - \int h(\theta - D)I(\underline{x}|\theta)dF_n(\theta) < -c_0.$$

Then when $\epsilon < c_0$ and $n > N$ we see $D_n(\epsilon)$ cannot be greater than b . We need some definitions before we start.

Let $c \equiv \int_{-\infty}^{\infty} l(\underline{x}|\theta) dF_{\infty}(\theta)$; b is so large that $b > d$,
 $h(t_0 - b) > 2(h(t_1 - d) + h(t_0 - d) + 4)$ and the E_1 , which satisfies
 $d < E_1 < b$ and $h(E_1 - d) = h(E_1 - b)$, also satisfies $E_1 > t_0$ and
 $F_{\infty}(E_1) > 1 - \frac{P_{\infty}(t_0) - P_{\infty}(t_1)}{8} \cdot r \cdot c$. Without loss of generality, we
 can assume E_1 is a continuous point of $F_{\infty}(\theta)$.

In our problem, we can see $D_0 > t_0$ (or we can find $D = t_0 \geq D_0$
 such that the inequality (4.13) is not satisfied). And b is
 independent of $F_n(\theta)$.

We can find N_0 , such that for all $n > N_0$,
 $F_n(E_1) > 1 - \frac{P_{\infty}(t_0) - P_{\infty}(t_1)}{6} \cdot r \cdot c$, $P_n(t_0) - P_n(t_1) > \frac{P_{\infty}(t_0) - P_{\infty}(t_1)}{2}$ and
 $\int l(\underline{x}|\theta) dF_n(\theta) > \frac{c}{2}$.

Now for all $D > b$ and let $h(E - d) = h(E - D)$. Then $E > E_1 > t_0$.
 And when $n > N$, the following inequalities follow immediately.

$$\begin{aligned}
 & \int h(\theta - d) l(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - D) l(\underline{x}|\theta) dF_n(\theta) \\
 &= \int_{-\infty}^E (h(\theta - d) - h(\theta - D)) l(\underline{x}|\theta) dF_n(\theta) + \int_E^{\infty} (h(\theta - d) - h(\theta - D)) l(\underline{x}|\theta) dF_n(\theta) \\
 &\leq \left(\int l(\underline{x}|\theta) dF_n(\theta) \right) \int_{t_1}^{t_0} (h(\theta - d) - h(\theta - D)) dP_n(\theta) + \int_E^{\infty} \frac{h(t_0 - D)}{r} dF_n(\theta) \\
 &\leq \frac{c}{2} (h(t_0 - d) + h(t_1 - d) - h(t_0 - D)) \int_{t_1}^{t_0} dP_n(\theta) + \frac{h(t_0 - D)}{r} (1 - F_n(E)) \\
 &\leq (h(t_0 - d) + h(t_1 - d) - h(t_0 - D)) \cdot \frac{c}{2} \cdot \frac{P_{\infty}(t_0) - P_{\infty}(t_1)}{2} + \\
 &\quad \frac{h(t_0 - D)}{r} \cdot \frac{P_{\infty}(t_0) - P_{\infty}(t_1)}{6} \cdot r \cdot c \\
 &\leq \frac{c(h(t_0 - d) + h(t_1 - d))(P_{\infty}(t_0) - P_{\infty}(t_1))}{4} - \frac{c(P_{\infty}(t_0) - P_{\infty}(t_1))}{12} \cdot \\
 &\quad (3(h(t_1 - d) + h(t_0 - d) + 4)) \\
 &\leq -c(P_{\infty}(t_0) - P_{\infty}(t_1)) .
 \end{aligned}$$

Thus $c_0 = c \cdot (P_\infty(t_0) - P_\infty(t_1))$ is what we wanted. And (a) follows directly.

(b) In this part, we have to construct $F_n \xrightarrow{W} F_\infty$ such that for all $D_\infty(\epsilon)$, (2.14) does not hold.

Let t_n be a continuous point of $P_\infty(\theta)$ satisfying $0 < P_\infty(t_n) < \frac{1}{n}$ (if there exists T such that $P_\infty(T) > 0$ and $P_\infty(T^-) = 0$, then let $T < t_n < T + \frac{1}{n}$ and t_n is a continuous point of $P_\infty(\theta)$).

Let $r_n = \frac{1}{2n^2}$ and $d = D_s(\epsilon)$, we can find $D_n > D_s(\epsilon)$ and θ_n

such that $\frac{h(t_n - D_n)}{(h(\theta_n - D_s(\epsilon)) - h(\theta_n - D_n))I(\underline{x}|\theta_n)} < \frac{1}{n^2}$, where

$h(E_n - D_s(\epsilon)) = h(E_n - D_n)$ and $\theta_n > E_n$.

We know t_n , as a sequence, is monotonic decreasing, so

$\lim_{n \rightarrow \infty} t_n$ exists. Let $T = \lim_{n \rightarrow \infty} t_n$, then $T = -\infty$ or T is finite.

We have to construct $F_n(\theta)$ under these two situations.

(i) if $T = -\infty$, then let

$$F_n(\theta) = (1 - \frac{1}{n})F_n(\theta) \cdot I(\theta \geq t_n) + \frac{1}{n} J_n(\theta)$$

where I is the usual indicator function, and

$$J_n(\theta) = \begin{cases} 0 & \theta < \theta_n \\ 1 & \theta \geq \theta_n \end{cases}.$$

(ii) if $T \neq -\infty$, the above $F_n(\theta)$ does not necessarily converge in distribution to $F_\infty(\theta)$. This will occur, for example, if $F_\infty(\theta)$ puts some probability on θ which has likelihood value that equals zero.) So in this case we define $F_n(\theta)$ as follows:

$$F_n(\theta) = \begin{cases} (1 - \frac{1}{n})F_\infty(\theta) & \theta < T \\ (1 - \frac{1}{n})F_\infty(\theta) - I(\theta \geq t_n) + \frac{1}{n} J_n(\theta) + \\ (1 - \frac{1}{n})F_\infty(\theta)I(\theta < T) & \theta \geq T. \end{cases}$$

Then

$$\begin{aligned} & \int h(\theta - D_\infty(\epsilon)) \ell(\underline{x}|\theta) dF_n(\theta) - \int h(\theta - D_n) \ell(\underline{x}|\theta) dF_n(\theta) \\ & \geq (1 - \frac{1}{n}) (h(t_n - D_\infty(\epsilon)) - h(t_n - D_n)) \cdot \ell(\underline{x}|t_n) F_\infty(t_n) + \\ & (1 - \frac{1}{n}) \left(\int_{t_n}^{\infty} h(\theta - D_\infty(\epsilon)) \ell(\underline{x}|\theta) dF_\infty(\theta) - \int_{t_n}^{\infty} h(\theta - D_n) \ell(\underline{x}|\theta) dF_\infty(\theta) \right) + \\ & n^2 h(t_n - D_n) \cdot \frac{1}{n} \\ & \geq -(1 - \frac{1}{n}) h(t_n - D_n) \cdot B + (1 - \frac{1}{n}) \int_{t_n}^{D_n} (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n)) \\ & \ell(\underline{x}|\theta) dF_\infty(\theta) + n \cdot h(t_n - D_n). \end{aligned}$$

(Since $\ell(\underline{x}|\theta) \leq B$ and when $\theta \in (D_n, \infty)$ we know $(h(\theta - D_\infty(\epsilon)) - h(\theta - D_n)) > 0$.)

$$\begin{aligned} & \geq -Bh(t_n - D_n) - Bh(t_n - D_n) + nh(t_n - D_n) \\ & \rightarrow +\infty. \end{aligned}$$

Thus the triple is unstable. Q.E.D.

The following theorem discusses the conditions that $D_n(\epsilon)$ will not go to $-\infty$. We state it without giving any proof. The proof should be symmetrically related to the proof of the last theorem.

Theorem 4.4 (a) Suppose there exist $r > 0$, d , t_0 and D_0 , where t_0 is a continuous point of $P_\infty(\theta)$ such that $P_\infty(t_0) < 1$, and for all $D < D_0$,

$$\frac{h(t_0 - D)}{\sup_{\theta < E} (h(\theta - d) - h(\theta - D)) \mathcal{L}(\underline{x}|\theta)} > r$$

where $D < E < d$ and $h(E - d) = h(E - D)$. Then $D_n(\epsilon)$ cannot go to $-\infty$.

(b) Suppose for all $r > 0$, d , t and D_0 , where t is a continuous point of $P_\infty(\theta)$ and $P_\infty(t) < 1$, and there exists $D < D_0$ such that

$$\frac{h(t - D)}{\sup_{\theta > E} (h(\theta - d) - h(\theta - D)) \mathcal{L}(\underline{x}|\theta)} < r$$

where $D < E < d$ and $h(E - d) = h(E - D)$.

Then the triple $(h, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable.

Proof: Similar to Theorem 4.3. Q.E.D.

In the following, we show an example that applies the above theorems.

Example 4.2 (a) Suppose $|\theta|^{p-1} \mathcal{L}(\underline{x}|\theta)$ is uniformly bounded and $p \geq 1$, then $(|\theta - D|^p, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is strongly stable.

(b) Suppose $|\theta|^{p-1} \mathcal{L}(\underline{x}|\theta)$ is unbounded and $p \geq 1$, then $(|\theta - D|^p, \mathcal{L}(\underline{x}|\theta), F_\infty)$ is unstable.

Proof: (a) By Theorems 4.2, 4.3 and 4.4, in order to show strongly stable we have to show (i) For any $D_\infty(\epsilon)$ and D , $(h(\theta - D_\infty(\epsilon)) - h(\theta - D)) \cdot \mathcal{L}(\underline{x}|\theta)$ as a function of θ , is uniformly bounded above, and (ii) there exists r , d , t_0 , D_0 and $P_\infty(t_0) > 0$ such that for all $D > D_0$,

$$\frac{h(t_0-D)}{\sup_{\theta > E} (h(\theta-d) - h(\theta-D))} \ell(\underline{x}|\theta) > r$$

(i) By assumption, let $|\theta|^{p-1} \ell(\underline{x}|\theta) \leq B_0$.

First, we show for any D , $|\theta-D|^{p-1} \ell(\underline{x}|\theta)$ is uniformly bounded. When θ is small, the function is uniformly bounded, so we have to consider only when θ is large

$$\lim_{\theta \rightarrow +\infty} |\theta-D|^{p-1} \ell(\underline{x}|\theta) = \lim_{\theta \rightarrow +\infty} \left(\left| \frac{\theta-D}{\theta} \right|^{p-1} \right) \theta^{p-1} \ell(\underline{x}|\theta) = \lim_{\theta \rightarrow +\infty} \theta^{p-1} \ell(\underline{x}|\theta).$$

Thus the above formula is uniformly bounded. Then the same result holds when $\theta \rightarrow -\infty$.

Next we show for any two decisions D_1 and D_2 , $(|\theta-D_1|^p - |\theta-D_2|^p) \ell(\underline{x}|\theta)$ is uniformly bounded. We use the following inequality (Hardy, Littlewood and Polya, 1934).

If x and y are positive and unequal, then

$$r x^{r-1}(x-y) > x^r - y^r > r y^{r-1}(x-y) \quad (r > 1).$$

So we have:

$$p(|\theta-D_1|^{p-1})(|\theta-D_1| - |\theta-D_2|) > |\theta-D_1|^p - |\theta-D_2|^p > p|\theta-D_2|^{p-1}(|\theta-D_1| - |\theta-D_2|)$$

Since both $|\theta-D_1|^{p-1} \ell(\underline{x}|\theta)$ and $|\theta-D_2|^{p-1} \ell(\underline{x}|\theta)$ are uniformly bounded, we can see that $(|\theta-D_1|^p - |\theta-D_2|^p) \ell(\underline{x}|\theta)$ is uniformly bounded. And for any $D_\infty(\epsilon)$ and D , $(|\theta-D_\infty(\epsilon)|^p - |\theta-D|^p) \ell(\underline{x}|\theta)$ is uniformly bounded above.

(ii) We have to show that there exists d, t_0, r, D_0 where t_0 is a continuous point of $P_\infty(\theta)$ and $P_\infty(t_0) > 0$, such that for all $D > D_0$,

$$\frac{h(t_0 - D)}{\sup_{\theta > E} (h(\theta - d) - h(\theta - D)) \mathcal{L}(\underline{x}(\theta))} > r.$$

Let t_0 be any point that is a continuous point of $p_\infty(\theta)$ and $P_\infty(t_0) > 0$. And let $d = t_0$, $D_0 > d$ and satisfying

$$\left(\frac{D_0 + d}{2}\right)^{p-1} (1 - 2^{-p})^{-1} > B_0. \text{ We will specify } r \text{ later.}$$

If $h(\theta - D) = h(\theta - D)$, then $\theta = \frac{d+D}{2}$, thus $E = \frac{d+D}{2}$.

$$\text{Let } A = \frac{h(t_0 - D)}{\sup_{\theta > E} (h(\theta - d) - h(\theta - D)) \mathcal{L}(\underline{x}(\theta))} \geq \frac{|D-d|^p}{\sup_{\theta > \frac{d+D}{2}} B_0 \cdot \frac{|\theta-d|^{p-1} |\theta-D|^p}{\theta^{p-1}}}$$

We consider first $\theta \in (\frac{d+D}{2}, D)$, and then $\theta > D$.

$$(i)' \quad D \geq \theta > \frac{d+D}{2}$$

$$\sup_{D \geq \theta > \frac{d+D}{2}} B_0 \cdot \frac{|\theta-d|^{p-1} |\theta-D|^p}{\theta^{p-1}} \leq B_0 \frac{|D-d|^{p-1} \frac{d+D}{2} - D|^p}{|\frac{d+D}{2}|^{p-1}}$$

$$= B \cdot \frac{(1-2^{-p})(D-d)^p}{|\frac{d+D}{2}|^{p-1}}$$

$$\text{So } A \geq \frac{(D-d)^p}{B \cdot (1-2^{-p})(D-d)^p} = \frac{|\frac{d+D}{2}|^{p-1}}{B(1-2^{-p})} > 1$$

$$(ii)' \quad \theta > D$$

$$\sup_{\theta > D} \left(\frac{(\theta-d)^p - (\theta-D)^p}{\theta^{p-1}} \right) \leq \sup_{\theta > D} \left(\frac{p(\theta-d)^{p-1}(D-d)}{\theta^{p-1}} \right)$$

However $\sup_{\theta > D} \frac{(\theta-d)^{p-1}}{\theta^{p-1}}$ is uniformly bounded, say by B_1 ,

then $\frac{(D-d)^p}{\sup_{\theta > D} \frac{(\theta-d)^{p-1}}{\theta^{p-1}}} \geq \frac{\theta(D-d)^p}{B_1(D-d)} = \frac{(D-d)^{p-1}}{B_1}$, thus also bounded away from zero.

Thus $D_n(\epsilon)$ cannot go to $+\infty$.

Similarly, $D_n(\epsilon)$ cannot go to $-\infty$. And the triple $(h, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable.

(b) Let t_0 be any point satisfying $p_\infty(t_0) > 0$ and for all $d, r > 0$,

$$\frac{h(t-D)}{\sup_{\theta > E} (h(\theta-d) - h(\theta-D)) \ell(\underline{x}|\theta)} = \frac{(D-t)^p}{\sup_{\theta > E} (|\theta-d|^p - |\theta-D|^p) \ell(\underline{x}|\theta)}$$

However, $\lim_{\theta \rightarrow \infty} (|\theta-d|^p - |\theta-D|^p) \ell(\underline{x}|\theta) \geq \lim_{\theta \rightarrow \infty} p(D-d) \theta^{p-1} \ell(\underline{x}|\theta) = +\infty$

$$\text{Thus } \frac{(D-t)^p}{\sup_{\theta > E} (h(\theta-d) - h(\theta-D)) \ell(\underline{x}|\theta)} = 0 \quad \forall D.$$

And the triple $(|\theta-D|^p, \ell(\underline{x}|\theta), F_\infty)$ is unstable. ■

The above example shows explicitly that if we know likelihood function exactly then the data give us a lot of information. For any fixed \underline{x} , many likelihood functions satisfy the condition that $|\theta| \ell(\underline{x}|\theta)$ is uniformly bounded. Thus $(|\theta-D|^2, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable for these triples. However $(|\theta-D|^2, 1, F_\infty)$, where $\ell(\underline{x}|\theta) \equiv 1$, is unstable. Another point is that if our data is from a normal distribution with mean θ and variance 1, then for any p , $(|\theta-D|^p, \ell(\underline{x}|\theta), F_\infty)$ is strongly stable.

Chapter 5 Stable Decision Problems Related to $\mathcal{L}(\underline{x}|\theta) \equiv 1$

5.1 Review

In this chapter we return to the problems discussed in Kadane and Chuang [1978]. When $\mathcal{L}(\underline{x}|\theta) \equiv 1$, then definitions I, II, III and IV are exactly the same as definitions 1, 2, 3 and 4 respectively. In order to be consistent with the notation of Kadane and Chuang [1978], in this chapter we study stability of a pair (h, F_∞) under definitions 1, 2, 3 and 4. We study the estimation problem, i.e. $h(x)$ is continuous, non-increasing in $(-\infty, 0)$ and non-decreasing in $(0, \infty)$. In Chapter 3, we proved definitions I, II, III and IV are equivalent for estimation problems. Hence, this is also true for definitions 1, 2, 3 and 4. Without loss of generality, we use definition 3 and assume $h(0) = 0$ throughout this chapter.

As in Chapters 3 and 4, we first study $h(x)$ uniformly bounded, then $h(x)$ uniformly bounded on one side and unbounded on the other side. Finally, we study $h(x)$ unbounded on both sides.

Lemma 5.1 If $h(x)$ is uniformly bounded, then for any $F_\infty(\theta)$, $(h, F_\infty(\theta))$ is strongly stable.

Proof: This is just a special case of Theorem 3.2. ■

Lemma 5.2 If $h(x)$ is uniformly bounded on one side and unbounded on the other side, then for any $F_\infty(\theta)$, $(h, F_\infty(\theta))$ is unstable.

Proof: This is a special case of Theorem 4.1(b).

Now, we consider the case $\lim_{x \rightarrow +\infty} h(x) = +\infty$. Let $g(t) = h(t+1) - h(t)$. ■

Lemma 5.3 If $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $g(t)$ is not uniformly bounded, then for any $F_\infty(\theta)$, $(h, F_\infty(\theta))$ is unstable.

Proof: Suppose $g(t)$ is not uniformly bounded. For any $F_\infty(\theta)$ let $\{D_\infty(\epsilon)\}$ be the set of all decisions satisfying (1.2). Also let $D_s(\epsilon) = \sup\{D_\infty(\epsilon)\}$, and $D_0 = D_s(\epsilon) + 1$ and $\theta' = \theta - D_0$.

Then $(h(\theta - D_s(\epsilon)) - h(\theta - D_0)) = h(\theta' + 1) - h(\theta')$, which is not uniformly bounded above. Thus by Lemma 4.2 (a), the pair $(h, F_\infty(\theta))$ is unstable.

Similarly, if $g(t)$ is not uniformly bounded below, by Lemma 4.2 (b), the pair is unstable. ■

One example for the above lemma is square error loss function with any opinion that has finite variance. Since $(x+1)^2 - x^2 = 2x+1$, which is not uniformly bounded, $(|\theta - D|^2, F_\infty(\theta))$ is unstable.

The case that is left is $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $g(t) = h(t+1) - h(t)$ is uniformly bounded. Let $|g(t)| \leq B$.

Lemma 5.4 If $g(t)$ is uniformly bounded by a constant B , then for any x and y , $|h(x) - h(y)| < B(|x-y| + 1)$.

Proof: Because the formula we want to show is symmetric with respect to x and y , i.e. exchange x and y will not change the formula, we can assume $x > y$.

Let $[x]$ denote the maximum integer which is smaller than or equal to x .

Suppose n is a positive integer. Then

$$\begin{aligned} h(n) &= h(n) - h(n-1) + h(n-1) - h(n-2) + \dots + h(2) - h(1) + h(1) - h(0) \\ &\leq n \cdot B. \end{aligned}$$

If $t > 0$ and t is not an integer, then $t < [t] + 1$ and $h(t) \leq h([t]+1) \leq B([t]+1) \leq B(t+1)$.

Thus, for all $u \geq 0$, $h(u) \leq B(u+1)$.

Similarly, for all $v \leq 0$, $h(v) \leq B(|v| + 1)$.

Now if $x > y > 0$, let $z = [x-y] + y + 1$ then $z \geq x$ and
 $|h(x) - h(y)| \leq |h(z) - h(y)| \leq B([x-y] + 1) \leq B(|x-y| + 1)$.

If $x \geq 0 \geq y$, then

$$|h(x) - h(y)| \leq \max(h(x), h(y)) \leq \max B \cdot (x+1, |y| + 1) \leq B(|x-y| + 1)$$

If $0 \geq x \geq y$, the proof is the same as $x \geq y \geq 0$. Q.E.D.

Lemma 5.5 Suppose $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $|g(t)| \leq B$. If there exist a, b and N such that for all $n > N$, $D_n \in [a, b]$, then (h, F_∞) is strongly stable; otherwise, the pair is unstable and there exists $F_n \rightarrow F_\infty$ such that

$$\lim_{n \rightarrow \infty} \left(\int h(\theta - D_\infty(\epsilon)) dF_n(\theta) - \inf_D \int h(\theta - D) dF_n(\theta) \right) = +\infty.$$

Proof: The first part is a direct result of Lemmas 5.4 and 4.3.

For the second part, by assumption, there exists $H_n \xrightarrow{W} F_\infty$ and a sequence of ϵ -optimal decisions of (h, H_n) , $D'_n(\epsilon)$ which goes to infinity. Without loss of generality we assume $D'_n(\epsilon) \rightarrow +\infty$. Let $D_s(\epsilon) = \sup\{D_\infty(\epsilon)\}$, where $\{D_\infty(\epsilon)\}$ is the set of all $D_\infty(\epsilon)$. Then $h(D_n(\epsilon) - D_s(\epsilon)) \rightarrow +\infty$. Thus we can find a subsequence of H_n , we call it G_n , such that $G_n \rightarrow F_\infty$ and $h(D_n(\epsilon) - D_s(\epsilon)) > n^2$, where $D_n(\epsilon)$ is an ϵ -optimal decision of (h, G_n) .

Let $F_n(\theta) = \frac{1}{n} J_n(\theta) + (1 - \frac{1}{n}) G_n(\theta)$, where $J_n(\theta)$ is defined as

$$J_n(\theta) = \begin{cases} 0 & \theta < D_n(\epsilon) \\ 1 & \theta \geq D_n(\epsilon) \end{cases}.$$

Then,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \left(\int h(\theta - D_\infty(\epsilon)) dF_n(\theta) - \inf_D \int h(\theta - D) dF_n(\theta) \right) \\
 & \geq \lim_{n \rightarrow \infty} \left(\left(1 - \frac{1}{n}\right) \left(\int h(\theta - D_\infty(\epsilon)) dG_n(\theta) - \int h(\theta - D_\infty(\epsilon)) dG_n(\theta) - \right. \right. \\
 & \quad \left. \left. \int h(\theta - D_n(\epsilon)) dG_n(\theta) \right) + \frac{1}{n} \cdot h(D_n(\epsilon) - D_\infty(\epsilon)) \right) \\
 & \geq \lim_{n \rightarrow \infty} \left(\left(1 - \frac{1}{n}\right) \cdot (-\epsilon) + n \right) \\
 & \geq \lim_{n \rightarrow \infty} (n - \epsilon) \\
 & = +\infty. \quad \blacksquare
 \end{aligned}$$

5.2 Stability, Independent of Opinion

The object of this section is to show that for any two opinions $F_{\infty}^{(\theta)}$ and $G_{\infty}(\theta)$, $(h, F_{\infty}(\theta))$ is strongly stable iff $(h, G_{\infty}(\theta))$ is strongly stable. From Lemmas 5.1, 5.2 and 5.3, we see this is true for the following cases: (a) $h(x)$ is uniformly bounded (b) $h(x)$ is uniformly bounded on one side and unbounded on the other side. (c) $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $g(t) = h(t+1) - h(t)$ is not uniformly

bounded. The loss functions we have not yet considered are those for which $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $g(t)$ is uniformly bounded. For this

kind of loss functions we prove our object by three lemmas. They are straightforward.

Lemma 5.6 Suppose $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $|g(t)|$ is uniformly bounded

by B . Also if we can find $a > -\infty$ and $b < +\infty$ such that $F_{\infty}(a) = G_{\infty}(a) = 0$ and $F_{\infty}(b) = G_{\infty}(b) = 1$, then $(h, F_{\infty}(\theta))$ is strongly stable iff $(h, G_{\infty}(\theta))$ is strongly stable.

Proof: If $(h, F_{\infty}(\theta))$ is unstable, we should show $(h, G_{\infty}(\theta))$ is also unstable.

By Lemma 5.5, there exists $F_n(\theta) \xrightarrow{W} F_{\infty}(\theta)$ such that $D_n(\epsilon) \rightarrow +\infty$ or $-\infty$. Without loss of generality, we assume $D_n(\epsilon) \rightarrow +\infty$. And we can assume $h(D_n(\epsilon) - b) > n^2$. (We always can achieve this by taking a subsequence of $F_n(\theta)$.)

And let $H_n(\theta) = (1 - \frac{1}{n}) F(\theta) + \frac{1}{n} J_n(\theta)$, where $J_n(\theta)$, is defined as

$$J_n(\theta) = \begin{cases} 0 & \theta < D_n(\epsilon) \\ 1 & \theta \geq D_n(\epsilon) \end{cases}.$$

Then for any $D \in [a, b]$,

$$\begin{aligned} & \int h(\theta - D) dH_n(\theta) - \int h(\theta - D_n(\epsilon)) dH_n(\theta) \\ &= (1 - \frac{1}{n}) (\int h(\theta - D) dF_n(\theta) - \int h(\theta - D_n(\epsilon)) dF_n(\theta)) + \frac{1}{n} \cdot h(D_n(\epsilon) - D) \\ &\geq \epsilon + n. \end{aligned}$$

Now we are ready to construct a sequence of distributions that converges in distributions to $G_\infty(\theta)$ and satisfies our object. Let

$$\begin{aligned} (5.1) \quad M_n &= (H_n(b) - H_n(a))(G_\infty(\theta)) + I(\theta \leq a) H_n(\theta) + I(\theta > a) H_n(a) \\ &\quad + I(\theta > b)(H_n(\theta) - H_n(b)) \\ \text{where } I(\theta > a) &= \begin{cases} 0 & \text{if } \theta \leq a \\ 1 & \text{if } \theta > a. \end{cases} \end{aligned}$$

We can see M_n is a distribution function, such that for all $\theta \in (-\infty, a) \cup (b, \infty)$, $M_n(\theta) = H_n(\theta)$, and has the same shape (up to a linear transformation) as $G_\infty(\theta)$ for $\theta \in [a, b]$. Since $H_n(b) \rightarrow 1$ and $H_n(a) \rightarrow 0$, we can see $M_n(\theta) \xrightarrow{w} G_\infty(\theta)$.

However, for all $D \in [a, b]$,

$$\begin{aligned}
& \int h(\theta-D) dM_n(\theta) - \int h(\theta-D_n(\epsilon)) dM_n(\theta) \\
&= \int_{a-}^{b+} (h(\theta-D) - h(\theta-D_n(\epsilon))) dM_n(\theta) + \int_{-\infty}^a + \int_b^{\infty} (h(\theta-D) - h(\theta-D_n(\epsilon))) dH_n(\theta) \\
&\geq [(\max_{\theta \in [a,b]} h(\theta-D) - B(b-a+1) - \min_{\theta \in [a,b]} h(\theta-D_n(\epsilon)) - B(b-a+1)] \\
& (H_n(b) - H_n(a)) + \int_{-\infty}^a + \int_b^{\infty} (h(\theta-D) - h(\theta-D_n(\epsilon))) dH_n(\theta) \\
&\geq \int_{-\infty}^{\infty} (h(\theta-D) - h(\theta-D_n(\epsilon))) dH_n(\theta) - 2B(b-a+1) \\
&\geq n - \epsilon - 2B(b-a+1) \rightarrow +\infty
\end{aligned}$$

Thus $(h, G_{\infty}(\theta))$ is also unstable.

In the next lemma, we extend the above lemma in such a way that one of the distributions is not restricted to $[a, b]$.

Lemma 5.7: Suppose $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $g(t)$ is uniformly bounded,

let $F_{\infty}(\theta)$ be a distribution such that we can find $a > -\infty$ and $b < +\infty$ and $F(a) = 0, F(b) = 1$. And let $G_{\infty}(\theta)$ be any distribution, then if $(h, F_{\infty}(\theta))$ is unstable implies $(h, G_{\infty}(\theta))$ is also unstable.

Proof: Let $G_n(\theta)$ be defined as

$$G_n(\theta) = \begin{cases} 0 & \theta < -n \\ G_{\infty}(\theta) & -n \leq \theta < n \\ 1 & n \leq \theta \end{cases}$$

We can find N_0 such that $-N_0 < a < b < N_0$, then for all $n > N_0$, $G_n(-n-1) = F_\infty(-n-1) = 0$, $G_n(n+1) = F_\infty(n+1) = 1$. Thus $(h, G_n(\theta))$ is unstable.

By inspect Lemma 5.6 and the way we construct $M_n(\theta)$, we can find a distribution $G_{nn}(\theta)$ such that $G_\infty(\theta) - \frac{1}{n} < G_{nn}(\theta) < G_\infty(\theta) + \frac{1}{n}$ for all $-n+1 < \theta < n-1$ and an ϵ -optimal decision for $(h, G_{nn}(\theta))$ is greater than n . We can see $G_{nn}(\theta) \xrightarrow{w} G_\infty(\theta)$ and its ϵ -optimal decision goes to $+\infty$.

Thus $(h, G_\infty(\theta))$ is unstable. Q.E.D.

After the above two lemmas, we still cannot conclude that for any two distributions $F_\infty(\theta)$ and $G_\infty(\theta)$, $(h, F_\infty(\theta))$ is strongly stable iff $(h, G_\infty(\theta))$ is strongly stable. More specifically, when $F_\infty(\theta)$ is a distribution such that there exists $a > -\infty$ and $b < +\infty$ and $F_\infty(a) = 0$, $F_\infty(b) = 1$ and $(h, F_\infty(\theta))$ is strongly stable, under this condition we can not conclude for any $G_\infty(\theta)$, $(h, G_\infty(\theta))$ is strongly stable. The following lemma fills this gap.

Lemma 5.8: Suppose $\lim_{x \rightarrow +\infty} h(x) = +\infty$, $g(t)$ is uniformly bounded, and there is a distribution $H_\infty(\theta)$ such that $(h, H_\infty(\theta))$ is unstable. Then there exists a distribution function $F_\infty(\theta)$, and numbers a and b $(-\infty < a < b < \infty)$, such that $F_\infty(a) = 0$, $F_\infty(b) = 1$ and (h, F_∞) is unstable.

Proof: Let an ϵ -optimal decision for $(h, H_\infty(\theta))$ be denoted by $D_\infty(\epsilon)$. Then we can find a, b such that all $D_\infty(\epsilon) \in [a, b]$, a, b are continuous points of $H_\infty(\theta)$ and $H_\infty(a) < \frac{1}{8}$ and $H_\infty(b) > 7/8$.

Since $(h(\theta-D), H_\infty)$ is unstable, we can find $H_1, H_2, \dots \xrightarrow{W} H_\infty$ and a sequence of ϵ -optimal decisions $D_1(\epsilon), D_2(\epsilon), \dots$ for $(h, H_n(\theta))$ such that $\lim_{n \rightarrow \infty} D_n(\epsilon) = +\infty$, and $\lim_{n \rightarrow +\infty} \int h(\theta-D_\infty(\epsilon)) - h(\theta-D_n(\epsilon)) dH_n(\theta) = +\infty$, using Lemma 5.5.

Let $D_\infty(\epsilon)$ be any fixed ϵ -optimal decision for $(h, H_\infty(\theta))$.

Suppose E_n satisfies $h(E_n - D_\infty(\epsilon)) = h(E_n - D_n(\epsilon))$ and $D_\infty(\epsilon) \leq E_n \leq D_n(\epsilon)$.

Then as $D_n(\epsilon) \rightarrow +\infty$, we see $E_n \rightarrow +\infty$.

$$\text{Now let } F_\infty(\theta) = \begin{cases} 0 & \theta < a \\ \frac{1}{H(b)-H(a)} (H(\theta) - H(a)) & a \leq \theta < b \\ 1 & \theta \geq b. \end{cases}$$

And define $F_n(\theta)$ as follows:

$$F_n(\theta) = \begin{cases} \frac{2H_n(E_n)-1}{H_n(b)-H_n(a)} (H_n(\theta) - H_n(a)) & a \leq \theta < b \\ 2H_n(E_n)-1 & b \leq \theta < E_n \\ 2H_n(\theta)-1 & \theta \geq E_n \end{cases}$$

Then $F_n(\theta) \xrightarrow{W} F_\infty(\theta)$, and there exists N such that for all $n > N$, $H_n(b) > 6/8$, $H_n(a) < 2/8$ and $\int_a^b (h(\theta-D_\infty(\epsilon)) - h(\theta-D_n(\epsilon))) dH_n(\theta) < 0$.

Let $D \in [a, b]$,

$$\begin{aligned}
 & \int (h(\theta - D) - h(\theta - D_n(\epsilon))) dF_n(\theta) \\
 & \geq \int (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dF_n(\theta) - B(|D_\infty(\epsilon) - D| + 1) \quad (\text{by Lemma 5.4}) \\
 & \geq \frac{2H_n(E_n) - 1}{H_n(b) - H_n(a)} \int_a^b (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dH_n(\theta) + \\
 & \quad 2 \int_{E_n}^\infty (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dH_n(\theta) - B(b - a + 1) \\
 & \geq \frac{1}{\frac{6}{8} - \frac{2}{8}} \int (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dH_n(\theta) + \\
 & \quad 2 \int_{C_n}^\infty (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dH_n(\theta) - B(b - a + 1) \\
 & = 2 \left(\int_a^b + \int_{E_n}^\infty (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dH_n(\theta) \right) - B(b - a + 1) \\
 & \geq 2 \int_{-\infty}^\infty (h(\theta - D_\infty(\epsilon)) - h(\theta - D_n(\epsilon))) dH_n(\theta) - B(b - a + 1) \\
 & \rightarrow +\infty
 \end{aligned}$$

So (h, F_∞) is unstable. ■

Combining Lemmas 5.6, 5.7 and 5.8, we have the following theorem.

Theorem 5.1 Suppose $\lim_{x \rightarrow +\infty} h(x) = +\infty$, $g(t)$ is uniformly bounded

and $F_\infty(\theta)$ and $G_\infty(\theta)$ are any two distributions. Then $(h, F_\infty(\theta))$ is strongly stable iff $(h, G_\infty(\theta))$ is strongly stable.

Theorem 5.2 For any estimation or prediction loss function h , and any opinions $F_{\infty}(\theta)$ and $H_{\infty}(\theta)$, $(h, F_{\infty}(\theta))$ is strongly stable iff $(h, H_{\infty}(\theta))$ is strongly stable.

Proof: Combines Theorem 5.1 and Lemmas 5.1, 5.2 and 5.3. ■

5.3 Necessary and Sufficient Conditions for Stability

In this section we prove necessary and sufficient conditions for a pair $(h, F_\infty(\theta))$ to be stable, where $\lim_{x \rightarrow +\infty} h(x) = +\infty$ and $g(x)$ is uniformly bounded.

Theorem 5.3 (a) Suppose there exists r and $D_0 > 0$, such that for all $D > D_0$.

$$\frac{h(-D)}{\sup_{a>0} (h(a+D) - h(a))} > r ,$$

and for all $D < -D_0$

$$\frac{h(D)}{\sup_{a<0} (h(a+D) - h(a))} > r ,$$

then $(h, F_\infty(\theta))$ is strongly stable.

(b) If for all r and $D_0 > 0$, there exists $D > D_0$, such that

$$\sup_{a>0} \frac{h(-D)}{h(a+D) - h(a)} < r$$

or there exist $D < -D_0$

$$\frac{h(D)}{\sup_{a < 0} (h(a+D) - h(a))} < r ,$$

then $(h, F_{\infty}(\theta))$ is unstable.

Proof (a) We are going to use Theorem 4.3 to prove this theorem.

By theorem 5.1, it is sufficient to show $(h, F_{\infty}(\theta))$ is strongly stable for a specific distribution. Let

$$F_{\infty}(\theta) = \begin{cases} \frac{1}{2} & -1 < \theta < 1 \\ 0 & \text{otherwise} \end{cases}$$

And let $d = 0$, we see $(h, F_{\infty}(\theta))$ satisfies the conditions in Theorem 4.3 (a). Thus $D_n(\epsilon)$ can not go to $+\infty$.

Similary, $D_n(\epsilon)$ cannot go to $-\infty$.

By Lemma 5.5, we conclude $(h, F_{\infty}(\theta))$ is strongly stable.

(b) for this part, we should show for all $r, d, t, D_0 > 0$, there exist $D > D_0$ such that

$$\frac{h(t-D)}{\sup_{\theta > D} (h(\theta-d) - h(\theta-D)) \underline{h}(\underline{x}(\theta))} < r .$$

Let r_1 and D_1 satisfy $4r_1 < r$, $r_1 < 1$, and $D_1 > 0$, $h(-D_1) > 2B(|d|+|t|+2)$ and $h(D_1) > 2B(|d|+|t|+2)$.

By our assumption, we can find $D > D_1$ such that

$$\frac{h(-D)}{\sup_{a > 0} (h(a+D) - h(a))} < r_1 .$$

$$\text{Then } \frac{h(t-D)}{\sup_{\theta>0} (h(\theta-d)-h(\theta-D))} = \frac{h(t-D)}{\sup_{a>0} h(a+D-d)-h(a)} \quad (\text{let } a = \theta - D)$$

Let $A = \sup_{a>0} (h(a+D)-h(a))$, then $h(-D) < r_1 A$ and

$$B(|t|+1) < h(-D) < r_1 A.$$

Thus

$$\begin{aligned} \frac{h(t-D)}{\sup_{a>0} (h(a+D-d)-h(a))} &\leq \frac{h(-D) + B(|t|+1)}{\sup_{a>0} (h(a+D)-h(a)) - B(|d|+1)} \quad \text{by Lemma 5.4} \\ &< \frac{r_1 A + r_1 A}{A - \frac{r_1}{2} A} \\ &< 4r_1 < r. \end{aligned}$$

Thus (h, F_∞) when h satisfying the first part of (b). Similarly, for the second part (h, F_∞) is also unstable.

Although in this theorem, we assume $\lim_{x \rightarrow I_\infty} h(x) = +\infty$

and $g(t)$ is uniformly bounded. However, this is not necessary. In the estimation or prediction problem, the above necessary and sufficient conditions are true for every pair, except those that equal zero on one side and are uniformly bounded on the other side. However, the conditions in Section 5.1 are much easier to check.

5.4 Examples

From Theorem 5.2, we know in the estimation or prediction problem, whether a pair $(h, F_\infty(\theta))$ is stable or not depends on h only. In this section we show some important examples. Many of these examples were discussed in Kadane and Chuang [1978].

Example 5.1: Let $h(x) = \begin{cases} |x| & \text{if } -1 < x \\ 1 & \text{if } x \leq -1 \end{cases}$.

This loss function is bounded on one side and unbounded on the other side, by Lemma 5.2, so $(h, F_\infty(\theta))$ is unstable for any $F_\infty(\theta)$ which has finite mean. ■

Example 5.2: If $h(x) = x^2$, i.e. the usual square loss. Then $(h, F_\infty(\theta))$ is unstable for all $F_\infty(\theta)$ that has a finite variance.

Proof: $\lim_{x \rightarrow +\infty} h(x) = +\infty$. $g(t) = h(t+1) - h(t) = 2t+1$.

So $g(t)$ is not uniformly bounded, thus by Lemma 5.3 $(h, F_\infty(\theta))$ is unstable. ■

Example 5.3 If $|g(t)| < B$ and there exist $r > 0$ and $x_0 > 0$ such that for all $|x| > x_0$ we have $h(x) \geq r|x|$, then $(h, F_\infty(\theta))$ is strongly stable.

Proof: Let $D_0 = \max(x_0, 1)$, then for all $D > D_0$,

$$\frac{h(-D)}{\sup_{a>0} (h(a+D) - h(a))} \geq \frac{r|D|}{B(|D|+1)} \geq \frac{r_0}{2B}.$$

Thus $(h, F_\infty(\theta))$ satisfies the first part of Theorem 5.2 (a). Similarly it satisfies the second part of Theorem 5.2 (a). Thus $(h, F_\infty(\theta))$ is strongly stable. ■

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One special case of this example is $h(\theta, D) = a(\theta - D)I(\theta \geq D) + b(D - \theta)I(\theta < D)$, $a > 0$, $b > 0$, where I is the usual indicator function. Then $(h, F_\infty(\theta))$ is strongly stable. When $a = b$, this specializes to absolute error.

From the above two examples, we know squared error loss is unstable with any opinion that has finite mean and variance. However, absolute error loss with any opinion that has finite mean is strongly stable.

The next example shows that even if $|g(t)| < B$, $h(x)$ is symmetric, and $g(t)$ goes to zero as t goes to infinity, the pair may not be stable.

Example 5.4 Let $h(x)$ be symmetric with respect to 0 and

$$h(x) = \begin{cases} x & 0 \leq x < 1 \\ 1 & 1 < x \leq 2 \cdot 2^3 \\ (x - 2 \cdot j^{j+1}) \cdot \frac{1}{j} + (j-1)^{j-1} & 2j^{j+1} < x \leq 3j^{j+1} - j(j-1)^{j-1} \\ j^j & 3j^{j+1} - j(j-1)^{j-1} < x \leq 2(j+1)^{j+2} \end{cases}$$

Then $(h, F_\infty(\theta))$ is unstable.

Proof: Let $d_n = 3n^{n+1} - n(n-1)^{n-1} - 2n^{n+1} = n^{n+1} - n(n-1)^{n-1}$

$$a_n = 2 \cdot n^{n+1}.$$

Then $h(-d_n) = (n-1)^{n-1}$, $h(a_n + d_n) = n^n$ and $h(a_n) = (n-1)^{n-1}$.

$$\frac{h(-d_n)}{\sup_{a>0} (h(a+d_n) - h(a))} \leq \frac{h(-d_n)}{h(a_n + d_n) - h(a_n)} = \frac{(n-1)^{n-1}}{n^n - (n-1)^{n-1}} \rightarrow 0.$$

So $(h, F_\infty(\theta))$ is unstable. ■

The next two examples discuss loss functions that are concave or convex.

Example 5.5 If $h(x)$ is symmetric and concave, then $(h, F_\infty(\theta))$ is strongly stable.

Proof: We first show for any $x > 0, y > 0, h(x+y) \leq h(x) + h(y)$.

By concavity, we have

$$h(\alpha u + (1-\alpha)v) \geq \alpha h(u) + (1-\alpha)h(v), \quad 0 < \alpha < 1.$$

Let $u = x+y, r=0$ and $\alpha = x/(x+y)$ then

$$h(x) \geq \frac{x}{x+y} h(x+y)$$

$$(x+y)h(x) \geq xh(x+y).$$

Similarly $(x+y)h(y) \geq yh(x+y)$.

Summing the above inequalities, we have

$$(x+y)(h(x) + h(y)) \geq (x+y)h(x+y), \text{ so}$$

$$h(x) + h(y) \geq h(x+y).$$

Let $D > 0$. Then

$$\frac{h(-D)}{\sup_{a>0} (h(a+D) - h(a))} = \frac{h(-D)}{h(D)} = 1.$$

Similarly for $D < 0$, we have the same result.

Thus by Theorem 5.2, $(h, F_\infty(\theta))$ is strongly stable. ■

From the above example, we know $(|\theta - D|^p, F_\infty(\theta))$, where $0 < p \leq 1$, is strongly stable.

Example 5.6: Suppose $h(x)$ is convex and there exists $x_0 > 0$ such that $h(x_0) > 0$ and $h(-x_0) > 0$, then

- (a) If $|g(t)| < B$, then $(h, F_\infty(\theta))$ is strongly stable.
- (b) If $g(t)$ is unbounded, then $(h, F_\infty(\theta))$ is unstable.

Proof: (b) follows directly from Lemma 5.3.

(a) We first show that there exist $r_0 > 0$ such that

$$h(x) > rx \text{ for all } x > x_0.$$

For any $x > x_0$, let $x = \beta x_0$, $\beta > 1$ and $x_0 = \frac{1}{\beta} x$.

By convexity $h((1 - \frac{1}{\beta})0 + \frac{1}{\beta}x) \leq (1 - \frac{1}{\beta})h(0) + \frac{1}{\beta}h(x)$, so

$$\text{we have } h(x) \geq \beta h(x_0) = \frac{h(x_0)}{x_0} x.$$

$$\text{Similarly } x < -x_0, h(x) \geq \frac{h(x_0)}{x_0} |x|.$$

Let $r = \min(\frac{h(x_0)}{x_0}, \frac{h(-x_0)}{x_0})$, then by Example 3, $(h, F_\infty(\theta))$ is strongly stable. ■

5.5 Some Relations Between the Set of Definitions I, II, III and IV and the Set of Definitions 1, 2, 3 and 4.

In this chapter, we assume $l(\underline{x}|\theta) \equiv 1$ and study the estimation problem. The necessary and sufficient conditions for a pair (as shown in this chapter) are much easier to check than the necessary and sufficient conditions for a general triple (as shown in Chapter 4). The following theorem shows in some cases we don't have to use the necessary and sufficient conditions in Chapter 4. Definitions 1, 2, 3 and 4 are equivalent; the same is also true for definitions I, II, III and IV. Without loss of generality, we use definitions 3 and III in the following theorem.

Theorem 5.4 If (h, F_∞) is strongly stable by definition 3 then for any likelihood function $l(\underline{x}|\theta)$, the triple $(h, l(\underline{x}|\theta), F_\infty)$ is strongly stable by definition III.

Proof: For any $F_n \xrightarrow{W} F_\infty$, let $P_n(\theta)$ ($P_\infty(\theta)$) denote the posterior distribution of θ corresponding to the prior distribution $F_n(\theta)$ ($F_\infty(\theta)$) and the likelihood function $l(\underline{x}|\theta)$.

Because (h, F_∞) is strongly stable by definition 3 then, by Theorem 5.2, (h, P_∞) is also strongly stable by definition 3. Thus for any $H_n \xrightarrow{W} P_\infty$,

$$(5.2) \quad \lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} \left(\int h(\theta - D_\infty(\epsilon)) dH_n(\theta) - \inf_D \int h(\theta - D) dH_n(\theta) \right) = 0.$$

By Theorem 2.2, we know $P_n \xrightarrow{W} P_\infty$. Thus (5.2) is true for $P_n(\theta)$, and the triple $(h, l(\underline{x}|\theta), F_\infty)$ is strongly stable by definition III. ■

Thus for the estimation or prediction problem, if we know a pair (h, F_∞) is strongly stable by definitions 1, 2, 3 or 4 then for any likelihood function $l(\underline{x}|\theta)$, the triple $(h, l(\underline{x}|\theta), F_\infty)$ is also strongly stable by definitions I, II, III or IV. However, the converse is not necessarily true.

Chapter 6 Conclusions and Topics for Further Research

This thesis studies the influence of small variations in both loss function and prior opinion on the one dimensional estimation or prediction problem. There are many ways to define small variations in loss function and prior opinion. In Kadane and Chuang [1978], we started with weak convergence in distribution and uniformly convergence in both arguments in the loss function. This thesis is a continuation of that paper. In Chapter 2 to Chapter 4, we assume likelihood function is fixed and agreed upon and study stability problems. Necessary and sufficient conditions for a triple to be stable are available. In Chapter 5, we study the case $I(x|\theta) = 1$, i.e., the problem studied in Kadane and Chuang [1978]. Then whether a pair is stable or not depends on loss function only.

Some problems are interesting and need further research. The introduction of each different likelihood function is the same as introducing a different metric on opinions. In Chapter 2 to Chapter 4 we study the properties of these new metrics. We can also introduce new metrics on loss functions. Whether our results in Kadane and Chuang [1978] or in this thesis will still hold is an interesting question and needs further research.

In our problem, we only characterize a triple by strongly stable, weakly stable and unstable. When a triple is strongly stable, we still cannot tell quantitatively how good a decision is. More specifically, we do not know how large the $B_n(\epsilon)$ in (1.10) is. Further work is needed to develop an appropriate methodology for this problem.

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